

GLUING CONSTRUCTION OF COMPACT $\text{Spin}(7)$ -MANIFOLDS

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Abstract. We give a differential-geometric construction of compact manifolds with holonomy $\text{Spin}(7)$ which is based on Joyce's second construction of compact $\text{Spin}(7)$ -manifolds in [11] and Kovalev's gluing construction of G_2 -manifolds in [12]. We also give some examples of compact $\text{Spin}(7)$ -manifolds, at least one of which is *new*. Ingredients in our construction are *orbifold admissible pairs with a compatible antiholomorphic involution*. Here in this paper we need orbifold admissible pairs (\overline{X}, D) consisting of a four-dimensional compact Kähler orbifold \overline{X} with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and a smooth anticanonical divisor D on \overline{X} . Also, we need a compatible antiholomorphic involution σ on \overline{X} which fixes the singular points in \overline{X} and acts freely on the anticanonical divisor D . If two orbifold admissible pairs (\overline{X}_1, D_1) , (\overline{X}_2, D_2) with $\dim_{\mathbb{C}} \overline{X}_i = 4$ and compatible antiholomorphic involutions σ_i on \overline{X}_i satisfy the *gluing condition*, we can glue $(\overline{X}_1 \setminus D_1)/\langle \sigma_1 \rangle$ and $(\overline{X}_2 \setminus D_2)/\langle \sigma_2 \rangle$ together to obtain a compact Riemannian 8-manifold (M, g) whose holonomy group $\text{Hol}(g)$ is contained in $\text{Spin}(7)$. Furthermore, if the \hat{A} -genus of M equals 1, then M is a $\text{Spin}(7)$ -manifold, i.e., a compact Riemannian manifold with holonomy $\text{Spin}(7)$. We shall investigate our gluing construction using orbifold admissible pairs (\overline{X}_i, D_i) with $i = 1, 2$ when $D_1 = D_2 = D$ and D is a complete intersection in a weighted projective space, as well as when $(\overline{X}_1, D_1) = (\overline{X}_2, D_2)$ and $\sigma_1 = \sigma_2$ (the *doubling case*).

1. INTRODUCTION

The purpose of this paper is to give a gluing construction and examples of compact $\text{Spin}(7)$ -manifolds, i.e., compact Riemannian manifolds with holonomy $\text{Spin}(7)$. We constructed in our previous papers [6] and [7] Calabi-Yau threefolds and fourfolds by gluing together two asymptotically cylindrical Ricci-flat Kähler manifolds, using the gluing technique which Kovalev used in constructing compact G_2 -manifolds [12]. Such asymptotically cylindrical Ricci-flat Kähler manifolds X are obtained from admissible pairs (\overline{X}, D) by setting $X = \overline{X} \setminus D$. In the present paper we glue instead two asymptotically cylindrical $\text{Spin}(7)$ -orbifolds to construct a compact $\text{Spin}(7)$ -orbifold, and then resolve the singularities to obtain a compact $\text{Spin}(7)$ -manifold. Such asymptotically cylindrical $\text{Spin}(7)$ -orbifolds are obtained by setting $(\overline{X} \setminus D)/\langle \sigma \rangle$ from *orbifold admissible pairs* (\overline{X}, D) with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and a compatible antiholomorphic involution σ on \overline{X} . Originally, Joyce resolved $X = \overline{X} \setminus D$ to obtain compact $\text{Spin}(7)$ -manifolds when \overline{X} is a four-dimensional Calabi-Yau orbifold and $D = \emptyset$, so that $X = \overline{X}$ is *compact*: Beginning with a compact four-dimensional Calabi-Yau orbifold \overline{X} with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and an antiholomorphic involution σ on \overline{X} with $(\overline{X})^\sigma = \text{Sing } \overline{X}$, Joyce proved first that $Z = \overline{X}/\langle \sigma \rangle$ admits a torsion-free $\text{Spin}(7)$ -structure. Since the associated Riemannian metric is flat (Euclidean) around the singularities of Z , he then replaced the neighborhood of each singularity of Z with a suitable asymptotically locally Euclidean (ALE) $\text{Spin}(7)$ -manifold to obtain a family of simply-connected, smooth 8-manifolds $\{M^\epsilon\}$ for $\epsilon \in (0, 1]$ with a $\text{Spin}(7)$ -structure Φ^ϵ with *small torsion*, which satisfies $d\Phi^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ in a suitable sense. Finally, he proved that Φ^ϵ can be deformed to a torsion-free $\text{Spin}(7)$ -structure for sufficiently small ϵ using the analysis on $\text{Spin}(7)$ -structures, so that $M = M^\epsilon$ admits a Riemannian metric with holonomy $\text{Spin}(7)$. We note that asymptotically cylindrical $\text{Spin}(7)$ -manifolds are recently constructed by Kovalev in [13] by resolving $(\overline{X} \setminus D)/\langle \sigma \rangle$.

In our construction, we begin with two orbifold admissible pairs (\overline{X}_1, D_1) and (\overline{X}_2, D_2) , consisting of a compact Kähler orbifold \overline{X}_i with $\dim_{\mathbb{C}} \overline{X}_i = 4$ and a smooth anticanonical divisor D_i on \overline{X}_i . Also, we consider an antiholomorphic involution σ_i acting on each \overline{X}_i . As in Joyce's second construction of compact $\text{Spin}(7)$ -manifolds, we require that \overline{X}_i have isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and

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$(\overline{X}_i)^\sigma = \text{Sing } \overline{X}_i$ (see Definitions 3.6 and 3.11). In addition, we suppose that σ preserves and acts freely on D . Then by the existence result of an asymptotically cylindrical Ricci-flat Kähler form on $\overline{X}_i \setminus D_i$, each $\overline{X}_i \setminus D_i$ has a natural σ_i -invariant asymptotically cylindrical torsion-free $\text{Spin}(7)$ -structure, which pushes down to a torsion-free $\text{Spin}(7)$ -structure Φ_i on $\overline{X}_i / \langle \sigma_i \rangle$. Now suppose the *asymptotic models* $((D_i \times S^1) / \langle \sigma_{D_i \times S^1, \text{cyl}} \rangle \times \mathbb{R}_+, \Phi_{i, \text{cyl}})$ of $(\overline{X}_i \setminus D_i) / \langle \sigma_i \rangle, \Phi_i$ are isomorphic in a suitable sense, which is ensured by the *gluing condition* defined later (see Section 3.4.1). Then as in Kovalev's construction in [12], we can glue together $(\overline{X}_1 \setminus D_1) / \langle \sigma_1 \rangle$ and $(\overline{X}_2 \setminus D_2) / \langle \sigma_2 \rangle$ along their cylindrical ends $(D_1 \times S^1) / \langle \sigma_{D_1 \times S^1, \text{cyl}} \rangle \times (T-1, T+1)$ and $(D_2 \times S^1) / \langle \sigma_{D_2 \times S^1, \text{cyl}} \rangle \times (T-1, T+1)$, to obtain a compact 8-orbifold M_T^∇ . Also, we can glue together the torsion-free $\text{Spin}(7)$ -structures Φ_i on $(\overline{X}_i \setminus D_i) / \langle \sigma_i \rangle$ to construct a d-closed 4-form $\tilde{\Phi}_T^\nabla$ on M_T^∇ . Furthermore, replacing each neighborhood of singular points in M_T^∇ with a certain ALE $\text{Spin}(7)$ -manifold, we construct a family $(M_T^\epsilon, \tilde{\Phi}_T^\epsilon)$ of simply-connected, smooth 8-manifolds with a d-closed 4-form for sufficiently small $\epsilon > 0$, such that each $\tilde{\Phi}_T^\epsilon$ is projected to a $\text{Spin}(7)$ -structure $\Phi_T^\epsilon = \Theta(\tilde{\Phi}_T^\epsilon)$, with $\Phi_T^\epsilon \rightarrow 0$ as $T \rightarrow \infty$ or $\epsilon \rightarrow 0$ in a suitable sense. Now set $\epsilon = e^{-\gamma T}$ for some $\gamma > 0$, and $(M^\epsilon, \Phi^\epsilon) = (M_T^\epsilon, \tilde{\Phi}_T^\epsilon)$. Then using the analysis on $\text{Spin}(7)$ -structures by Joyce [11], we shall prove that Φ^ϵ can be deformed into a torsion-free $\text{Spin}(7)$ -structure for sufficiently small ϵ , so that the resulting compact manifold M^ϵ admits a Riemannian metric with holonomy contained in $\text{Spin}(7)$. Since $M = M^\epsilon$ is simply-connected, the \hat{A} -genus $\hat{A}(M)$ of M is 1, 2, 3 or 4, and the holonomy group is determined as $\text{Spin}(7), \text{SU}(4), \text{Sp}(2), \text{Sp}(1) \times \text{Sp}(1)$ respectively (see Theorem 2.8). Hence if $\hat{A}(M) = 1$, then M is a compact $\text{Spin}(7)$ -manifold.

Beginning with a Fano four-orbifold V with a smooth anticanonical divisor D , Kovalev obtained an orbifold admissible pair (\overline{X}, D) of Fano type as follows. Let S be a smooth complex surface in D representing the self-intersection class $D \cdot D$ on V . If we take \overline{X} to be the blow-up of V along S , then the proper transform of D in \overline{X} , which is isomorphic to D and denoted by D again, is an anticanonical divisor on \overline{X} with the holomorphic normal bundle $N_{D/\overline{X}}$ trivial. Hence (\overline{X}, D) is an orbifold admissible pair (Theorem 4.8).

For a given orbifold admissible pair (\overline{X}_1, D_1) with a compatible antiholomorphic involution σ_1 , it is difficult in general to find another admissible pair (\overline{X}_2, D_2) with σ_2 such that both $(\overline{X}_i \setminus D_i) / \langle \sigma_i \rangle$ have the same asymptotic model. One way to solve this is the ‘doubling’ method used in [6], [7], in which we take $(\overline{X}_1, D_1) = (\overline{X}_2, D_2)$ and $\sigma_1 = \sigma_2$. For another solution, we investigate orbifold admissible pairs (\overline{X}, D) of Fano type when V is a complete intersection in a weighted projective space $W = \mathbb{C}P^{k+3}(a_0, \dots, a_{k+3})$ with $k \geq 2$. Suppose σ is an antiholomorphic involution on W and

$$V = \{ [z] \in W \mid f_1(z) = \dots = f_{k-1}(z) = 0 \}, \quad D = \{ [z] \in W \mid f_1(z) = \dots = f_k(z) = 0 \},$$

where D is smooth and f_i are weighted homogeneous polynomials satisfying $\deg f_1 + \dots + \deg f_k = a_0 + \dots + a_{k+3}$ and $\sigma^* f_i = \overline{f_i}$ for $i = 1, \dots, k$. Then by the adjunction formula, V is a Fano four-orbifold with an anticanonical Calabi-Yau divisor D . Choosing f_{k+1} so that

$$\deg f_{k+1} = \deg f_k, \quad \sigma^* f_{k+1} = \overline{f_{k+1}} \quad \text{and} \\ S = \{ [z] \in W \mid f_1(z) = \dots = f_{k+1}(z) = 0 \} \quad \text{represents} \quad D \cdot D,$$

we have an orbifold admissible pair (\overline{X}_1, D_1) with a compatible antiholomorphic involution σ_1 such that $(D_1, \sigma_1|_{D_1})$ is isomorphic to $(D, \sigma|_D)$. Meanwhile, if we exchange f_k and f_{k-1} (and choose suitable f_{k+1} correspondingly), then V may change, but D does not change. Hence we have another orbifold admissible pair (\overline{X}_2, D_2) with σ_2 which has the same asymptotic model.

In the present paper, we shall give 3 topologically distinct compact $\text{Spin}(7)$ -manifolds, at least one of which is new. Each of the examples satisfies $b^2(M) = b^3(M) = 0$ and $\hat{A}(M) = 1$.

In order to show $\hat{A}(M) = 1$, we shall make an approach similar to [7], Section 4.4, that is, we reduce the problem to the computations on the cohomology groups of D and S . Using some results on weighted complete intersections in [8], we conclude that each resulting 8-manifold M satisfies $\hat{A}(M) = 1$. Betti numbers (b^2, b^3, b^4) of the compact $\text{Spin}(7)$ -manifolds in our construction are $(0, 0, 910)$, $(0, 0, 1294)$ and $(0, 0, 1678)$. Of these compact $\text{Spin}(7)$ -manifolds, the resulting manifold M with $\chi(M) = 1680$ is at least one new example which is not diffeomorphic to the known ones (see Theorem 4.10).

This paper is organized as follows. Section 2 is a brief review of $\text{Spin}(7)$ -structures. In Section 3 we define orbifold admissible pairs which will be ingredients in our gluing construction of compact $\text{Spin}(7)$ -manifolds.

This section is the heart of this paper. We consider compatible antiholomorphic involutions σ on orbifold admissible pairs (\bar{X}, D) and glue together two orbifold admissible pairs with $\dim_{\mathbb{C}} \bar{X} = 4$ divided by σ . The gluing theorems are stated in Section 3.5 including both cases of $\text{Spin}(7)$ -manifolds and Calabi-Yau fourfolds. Giving a quick review of basics on weighted projective spaces in Section 4.1, we obtain in Section 4.2 orbifold admissible pairs from complete intersections in weighted projective spaces. Then in Section 4.3 we give a new example of compact $\text{Spin}(7)$ -manifolds M with the Euler characteristic $\chi(M) = 1680$. We also find Betti numbers (b^2, b^3, b^4) of this $\text{Spin}(7)$ -manifold by applying the Mayer-Vietoris theorem. In the last section we shall give other examples of compact $\text{Spin}(7)$ -manifolds taking weighted complete intersections in $\mathbb{C}P^5(1, 1, 1, 1, 4, 4)$. All the resulting compact $\text{Spin}(7)$ -manifolds are listed in Table 5.4. Finally we illustrate an example of the doubling construction of Calabi-Yau fourfolds (Corollary 3.23) from orbifold admissible pairs.

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2. GEOMETRY OF $\text{Spin}(7)$ -STRUCTURES

Here we shall recall some basic facts about $\text{Spin}(7)$ -structures on oriented 8-manifolds. The material in this section is also discussed in [7], Section 2. For more details, see [11], Chapter 10.

We begin with the definition of $\text{Spin}(7)$ -structures on oriented vector spaces of dimension 8.

Definition 2.1. Let V be an oriented real vector space of dimension 8. Let $\{\theta^1, \dots, \theta^8\}$ be an oriented basis of V . Set

$$\begin{aligned} \Phi_0 &= \theta^{1234} + \theta^{1256} + \theta^{1278} + \theta^{1357} - \theta^{1368} - \theta^{1458} - \theta^{1467} \\ &\quad - \theta^{2358} - \theta^{2367} - \theta^{2457} + \theta^{2468} + \theta^{3456} + \theta^{3478} + \theta^{5678}, \\ g_0 &= \sum_{i=1}^8 \theta^i \otimes \theta^i, \end{aligned}$$

where $\theta^{ij\dots k} = \theta^i \wedge \theta^j \wedge \dots \wedge \theta^k$. Define the $\text{GL}_+(V)$ -orbit spaces

$$\begin{aligned} \mathcal{A}(V) &= \{ a^* \Phi_0 \mid a \in \text{GL}_+(V) \}, \\ \text{Met}(V) &= \{ a^* g_0 \mid a \in \text{GL}_+(V) \}. \end{aligned}$$

We call $\mathcal{A}(V)$ the set of *Cayley 4-forms* (or the set of $\text{Spin}(7)$ -structures) on V . On the other hand, $\text{Met}(V)$ is the set of positive-definite inner products on V , which is also a homogeneous space isomorphic to $\text{GL}_+(V)/\text{SO}(V)$, where $\text{SO}(V)$ is defined by

$$\text{SO}(V) = \{ a \in \text{GL}_+(V) \mid a^* g_0 = g_0 \}.$$

Now the group $\text{Spin}(7)$ is defined as the isotropy of the action of $\text{GL}(V)$ (in place of $\text{GL}_+(V)$) on $\mathcal{A}(V)$ at Φ_0 :

$$\text{Spin}(7) = \{ a \in \text{GL}(V) \mid a^* \Phi_0 = \Phi_0 \}.$$

Then one can show that $\text{Spin}(7)$ is a compact Lie group of dimension 27 which is a Lie subgroup of $\text{SO}(V)$ (see [9]). Thus we have a natural projection

$$\mathcal{A}(V) \cong \text{GL}_+(V)/\text{Spin}(7) \longrightarrow \text{GL}_+(V)/\text{SO}(V) \cong \text{Met}(V),$$

so that each Cayley 4-form (or $\text{Spin}(7)$ -structure) $\Phi \in \mathcal{A}(V)$ defines a positive-definite inner product $g_\Phi \in \text{Met}(V)$ on V .

Definition 2.2. Let V be an oriented vector space of dimension 8. If $\Phi \in \mathcal{A}(V)$, then we have the orthogonal decomposition

$$(2.1) \quad \wedge^4 V^* = T_\Phi \mathcal{A}(V) \oplus T_\Phi^\perp \mathcal{A}(V)$$

with respect to the induced inner product g_Φ . We define a neighborhood $\mathcal{T}(V)$ of $\mathcal{A}(V)$ in $\wedge^4 V^*$ by

$$\mathcal{T}(V) = \left\{ \Phi + \alpha \mid \Phi \in \mathcal{A}(V) \text{ and } \alpha \in T_\Phi^\perp \mathcal{A}(V) \text{ with } |\alpha|_{g_\Phi} < \rho \right\}.$$

We choose and fix a small constant ρ so that any $\chi \in \mathcal{T}(V)$ is uniquely written as $\chi = \Phi + \alpha$ with $\alpha \in T_{\Phi}^{\perp}\mathcal{A}(V)$. Thus we can define the projection

$$\Theta : \mathcal{T}(V) \longrightarrow \mathcal{A}(V), \quad \chi \longmapsto \Phi.$$

Lemma 2.3 (Joyce [11], Proposition 10.5.4). *Let $\Phi \in \mathcal{A}(V)$ and $\wedge^4 V^* = \wedge_+^4 V^* \oplus \wedge_-^4 V^*$ be the orthogonal decomposition with respect to g_{Φ} , where $\wedge_+^4 V^*$ (resp. $\wedge_-^4 V^*$) is the set of self-dual (resp. anti-self-dual) 4-forms on V . Then we have the following inclusion:*

$$\wedge_-^4 V^* \subset T_{\Phi}\mathcal{A}(V).$$

Now we define $\text{Spin}(7)$ -structures on oriented 8-manifolds.

Definition 2.4. Let M be an oriented 8-manifold. We define $\mathcal{A}(M) \longrightarrow M$ to be the fiber bundle whose fiber over x is $\mathcal{A}(T_x^*M) \subset \wedge^4 T_x^*M$. Then $\Phi \in C^{\infty}(\wedge^4 T^*M)$ is a *Cayley 4-form* or a *Spin(7)-structure* on M if $\Phi \in C^{\infty}(\mathcal{A}(M))$, i.e., Φ is a smooth section of $\mathcal{A}(M)$. If Φ is a Spin(7)-structure on M , then Φ induces a Riemannian metric g_{Φ} since $\Phi|_x$ for each $x \in M$ induces a positive-definite inner product $g_{\Phi}|_x$ on $T_x M$. A Spin(7)-structure Φ on M is said to be *torsion-free* if it is parallel with respect to the induced Riemannian metric g_{Φ} , i.e., $\nabla_{g_{\Phi}}\Phi = 0$, where $\nabla_{g_{\Phi}}$ is the Levi-Civita connection of g_{Φ} .

Definition 2.5. Let Φ be a Spin(7)-structure on an oriented 8-manifold M . We define $\mathcal{T}(M)$ to be the fiber bundle whose fiber over x is $\mathcal{T}(T_x^*M) \subset \wedge^4 T_x^*M$. Then for the constant ρ given in Definition 2.2, we have the well-defined projection $\Theta : \mathcal{A}(M) \longrightarrow \mathcal{T}(M)$. Also, we see from Lemma 2.3 that $\wedge_-^4 T^*M \subset T_{\Phi}\mathcal{A}(M)$ as subbundles of $\wedge^4 T^*M$.

Lemma 2.6 (Joyce, Proposition 10.5.9). *Let Φ be a Spin(7)-structure on M . There exist such that $\epsilon_1, \epsilon_2, \epsilon_3$ independent of M and Φ , such that the following is true.*

*If $\eta \in C^{\infty}(\wedge^4 T^*M)$ satisfies $\|\eta\|_{C^0} \leq \epsilon_1$, then $\Phi + \eta \in \mathcal{T}(M)$. For this η , $\Theta(\Phi + \eta)$ is well-defined and expanded as*

$$(2.2) \quad \Theta(\Phi + \eta) = \Phi + p(\eta) - F(\eta),$$

where $p(\eta)$ is the linear term and $F(\eta)$ is the higher order term in η , and for each $x \in M$, $p(\eta)|_x$ is the $T_{\Phi}\mathcal{A}(V)$ -component of $\eta|_x$ in the orthogonal decomposition (2.1) for $V = T_x^*M$. Also, we have the following pointwise estimates for any $\eta, \eta' \in C^{\infty}(\wedge^4 T^*M)$ with $|\eta|, |\eta'| \leq \epsilon_1$:

$$\begin{aligned} |F(\eta) - F(\eta')| &\leq \epsilon_2 |\eta - \eta'| (|\eta| + |\eta'|), \\ |\nabla(F(\eta) - F(\eta'))| &\leq \epsilon_3 \{ |\eta - \eta'| (|\eta| + |\eta'|) |d\Phi| + |\nabla(\eta - \eta')| (|\eta| + |\eta'|) \\ &\quad + |\eta - \eta'| (|\nabla\eta| + |\nabla\eta'|) \}. \end{aligned}$$

Here all norms are measured by g_{Φ} .

The following result is important in that it relates the holonomy contained in Spin(7) with the d-closedness of the Spin(7)-structure.

Theorem 2.7 (Salamon [16], Lemma 12.4). *Let M be an oriented 8-manifold. Let Φ be a Spin(7)-structure on M and g_{Φ} the induced Riemannian metric on M . Then the following conditions are equivalent.*

- (1) Φ is a torsion-free Spin(7)-structure, i.e., $\nabla_{g_{\Phi}}\Phi = 0$.
- (2) $d\Phi = 0$.
- (3) The holonomy group $\text{Hol}(g_{\Phi})$ of g_{Φ} is contained in Spin(7).

Now suppose $\tilde{\Phi} \in C^{\infty}(\mathcal{T}(M))$ with $d\tilde{\Phi} = 0$. We shall construct such a form $\tilde{\Phi}$ in Section 3.4.2. Then $\Phi = \Theta(\tilde{\Phi})$ is a Spin(7)-structure on M . If $\eta \in C^{\infty}(\wedge^4 T^*M)$ with $\|\eta\|_{C^0} \leq \epsilon_1$, then $\Theta(\Phi + \eta)$ is expanded as in (2.2). Setting $\phi = \tilde{\Phi} - \Phi$ and using $d\tilde{\Phi} = 0$, we have

$$d\Theta(\Phi + \eta) = -d\phi + dp(\eta) - dF(\eta).$$

Thus the equation $d\Theta(\Phi + \eta) = 0$ for $\Theta(\Phi + \eta)$ to be a torsion-free Spin(7)-structure is equivalent to

$$(2.3) \quad dp(\eta) = d\phi + dF(\eta).$$

In particular, we see from Lemma 2.3 that if $\eta \in C^{\infty}(\wedge_-^4 T^*M)$ then $p(\eta) = \eta$, so that equation (2.3) becomes

$$(2.4) \quad d\eta = d\phi + dF(\eta).$$

Joyce proved by using the iteration method and $dC^\infty(\wedge^4 T^*M) = dC^\infty(\wedge^4 T^*M)$ that equation (2.4) has a solution $\eta \in C^\infty(\wedge^4 T^*M)$ if ϕ is sufficiently small with respect to certain norms (see Theorem 3.26).

Theorem 2.8 (Joyce [11], Theorem 10.6.1). *Let (M, g) be a compact Riemannian 8-manifold such that its holonomy group $\text{Hol}(g)$ is contained in $\text{Spin}(7)$. Then the \hat{A} -genus $\hat{A}(M)$ of M satisfies*

$$(2.5) \quad 48\hat{A}(M) = 3\tau(M) - \chi(M),$$

where $\tau(M)$ and $\chi(M)$ is the signature and the Euler characteristic of M respectively. Moreover, if M is simply-connected, then $\hat{A}(M)$ is 1, 2, 3 or 4, and the holonomy group of (M, g) is determined as

$$\text{Hol}(g) = \begin{cases} \text{Spin}(7) & \text{if } \hat{A}(M) = 1, \\ \text{SU}(4) & \text{if } \hat{A}(M) = 2, \\ \text{Sp}(2) & \text{if } \hat{A}(M) = 3, \\ \text{Sp}(1) \times \text{Sp}(1) & \text{if } \hat{A}(M) = 4. \end{cases}$$

3. THE GLUING PROCEDURE

3.1. Compact complex manifolds with an anticanonical divisor. We suppose that \overline{X} is a compact complex manifold of dimension m , and D is a smooth irreducible anticanonical divisor on \overline{X} . We recall some results in [5], Sections 3.1–3.2, and [6], Sections 3.1–3.2.

Lemma 3.1. *Let \overline{X} and D be as above. Then there exists a local coordinate system $\{U_\alpha, (z_\alpha^1, \dots, z_\alpha^{m-1}, w_\alpha)\}$ on \overline{X} such that*

- (i) w_α is a local defining function of D on U_α , i.e., $D \cap U_\alpha = \{w_\alpha = 0\}$, and
- (ii) the m -forms $\Omega_\alpha = \frac{dw_\alpha}{w_\alpha} \wedge dz_\alpha^1 \wedge \dots \wedge dz_\alpha^{m-1}$ on $U_\alpha \setminus D$ together yield a holomorphic volume form Ω on $X = \overline{X} \setminus D$.

Next we shall see that $X = \overline{X} \setminus D$ is a cylindrical manifold whose structure is induced from the holomorphic normal bundle $N = N_{D/\overline{X}}$ to D in \overline{X} , where the definition of cylindrical manifolds is given as follows.

Definition 3.2. Let X be a noncompact differentiable manifold of dimension r . Then X is called a *cylindrical manifold* or a *manifold with a cylindrical end* if there exists a diffeomorphism $\pi : X \setminus X_0 \rightarrow \Sigma \times \mathbb{R}_+ = \{(p, t) \mid p \in \Sigma, 0 < t < \infty\}$ for some compact submanifold X_0 of dimension n with boundary $\Sigma = \partial X_0$. Also, extending t smoothly to X so that $t \leq 0$ on $X \setminus X_0$, we call t a *cylindrical parameter* on X .

Let (x_α, y_α) be local coordinates on $V_\alpha = U_\alpha \cap D$, such that x_α is the restriction of z_α to V_α and y_α is a coordinate in the fiber direction. Then one can see easily that $dx_\alpha^1 \wedge \dots \wedge dx_\alpha^{m-1}$ on V_α together yield a holomorphic volume form Ω_D , which is also called the *Poincaré residue* of Ω along D . Let $\|\cdot\|$ be the norm of a Hermitian bundle metric on N . We can define a cylindrical parameter t on N by $t = -\frac{1}{2} \log \|s\|^2$ for $s \in N \setminus D$. Then the local coordinates (z_α, w_α) on X are asymptotic to the local coordinates (x_α, y_α) on $N \setminus D$ in the following sense.

Lemma 3.3. *There exists a diffeomorphism Φ from a neighborhood V of the zero section of N containing $t^{-1}(\mathbb{R}_+)$ to a tubular neighborhood U of D in X such that Φ can be locally written as*

$$\begin{aligned} z_\alpha &= x_\alpha + O(|y_\alpha|^2) = x_\alpha + O(e^{-t}), \\ w_\alpha &= y_\alpha + O(|y_\alpha|^2) = y_\alpha + O(e^{-t}), \end{aligned}$$

where we multiply all z_α and w_α by a single constant to ensure $t^{-1}(\mathbb{R}_+) \subset V$ if necessary.

Hence X is a cylindrical manifold with the cylindrical parameter t via the diffeomorphism Φ given in the above lemma. In particular, when $H^0(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ and $N_{D/\overline{X}}$ is trivial, we have a useful coordinate system near D .

Lemma 3.4 ([6], Lemma 3.4.). *Let (\overline{X}, D) be as in Lemma 3.1. If $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ and the normal bundle $N_{D/\overline{X}}$ is holomorphically trivial, then there exists an open neighborhood U_D of D and a holomorphic function w on U_D such that w is a local defining function of D on U_D . Also, we may define the cylindrical parameter t with $t^{-1}(\mathbb{R}_+) \subset U_D$ by writing the fiber coordinate y of $N_{D/\overline{X}}$ as $y = \exp(-t - \sqrt{-1}\theta)$.*

3.2. Admissible pairs and asymptotically cylindrical Ricci-flat Kähler manifolds.

Definition 3.5. Let X be a cylindrical manifold such that $\pi : X \setminus X_0 \rightarrow \Sigma \times \mathbb{R}_+ = \{(p, t)\}$ is a corresponding diffeomorphism. If g_Σ is a Riemannian metric on Σ , then it defines a cylindrical metric $g_{\text{cyl}} = g_\Sigma + dt^2$ on $\Sigma \times \mathbb{R}_+$. Then a complete Riemannian metric g on X is said to be *asymptotically cylindrical* (to $(\Sigma \times \mathbb{R}_+, g_{\text{cyl}})$) if g satisfies for some cylindrical metric $g_{\text{cyl}} = g_\Sigma + dt^2$

$$\left| \nabla_{g_{\text{cyl}}}^j (g - g_{\text{cyl}}) \right|_{g_{\text{cyl}}} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{for all } j \geq 0,$$

where we regarded g_{cyl} as a Riemannian metric on $X \setminus X_0$ via the diffeomorphism π . Also, we call (X, g) an *asymptotically cylindrical manifold* and $(\Sigma \times \mathbb{R}_+, g_{\text{cyl}})$ the *asymptotic model* of (X, g) .

Definition 3.6. Let \overline{X} be a complex orbifold with isolated singular points $\text{Sing } \overline{X} = \{p_1, \dots, p_k\}$ and D a divisor on \overline{X} . Then (\overline{X}, D) is said to be an *orbifold admissible pair* if the following conditions hold:

- (a) \overline{X} is a compact Kähler orbifold.
- (b) D is a smooth anticanonical divisor on \overline{X} with $D \cap \text{Sing } \overline{X} = \emptyset$.
- (c) the normal bundle $N_{D/\overline{X}}$ is trivial.
- (d) \overline{X} and $\overline{X} \setminus (D \sqcup \text{Sing } \overline{X})$ are simply-connected.
- (e) Each $p \in \text{Sing } \overline{X}$ has a neighborhood U_p such that there exists a crepant resolution $\tilde{U}_p \dashrightarrow U_p$ at p .

Throughout this paper, we shall consider the action of \mathbb{Z}_4 on \mathbb{C}^4 generated by

$$(z_1, z_2, z_3, z_4) \mapsto (\sqrt{-1}z_1, \sqrt{-1}z_2, \sqrt{-1}z_3, \sqrt{-1}z_4) \quad \text{for } (z_1, z_2, z_3, z_4) \in \mathbb{C}^4.$$

If each U_p in condition (e) is isomorphic to $\mathbb{C}^4/\mathbb{Z}_4$, where the action of \mathbb{Z}_4 is given above, then we shall call (\overline{X}, D) an orbifold admissible pair with isolated singular points *modelled on* $\mathbb{C}^4/\mathbb{Z}_4$. This kind of orbifold admissible pair plays an important role later in constructing compact $\text{Spin}(7)$ -manifolds.

If \overline{X} is smooth, then $\text{Sing } \overline{X} = \emptyset$ and condition (e) is empty, so that the above conditions reduce to the definition of admissible pairs which originates in Kovalev [12] and is also used in our papers [6], [7]. From the above conditions, we see that Lemmas 3.1 and 3.4 apply to admissible pairs. Also, from conditions (a) and (b), we see that D is a compact Kähler manifold with trivial canonical bundle.

Theorem 3.7 (Tian-Yau [17], Kovalev [12], Hein [10]). *Let (\overline{X}, ω') be a compact Kähler manifold and $m = \dim_{\mathbb{C}} \overline{X}$. If (\overline{X}, D) is an admissible pair, then the following is true.*

It follows from Lemmas 3.1 and 3.4, there exist a local coordinate system $(U_{D,\alpha}, (z_\alpha^1, \dots, z_\alpha^{m-1}, w))$ on a neighborhood $U_D = \cup_\alpha U_{D,\alpha}$ of D and a holomorphic volume form Ω on $\overline{X} \setminus D$ such that

$$\Omega = \frac{dw}{w} \wedge dz_\alpha^1 \wedge \dots \wedge dz_\alpha^{m-1} \quad \text{on } U_{D,\alpha} \setminus D.$$

Let κ_D be the unique Ricci-flat Kähler form on D in the Kähler class $[\omega'|_D]$. Also let (x_α, y) be local coordinates of $N_{D/\overline{X}} \setminus D$ as in Section 3.1 and write y as $y = \exp(-t - \sqrt{-1}\theta)$. Now define a holomorphic volume form Ω_{cyl} and a cylindrical Ricci-flat Kähler form ω_{cyl} on $N_{D/\overline{X}} \setminus D$ by

$$(3.1) \quad \begin{aligned} \Omega_{\text{cyl}} &= \frac{dy}{y} \wedge dx_\alpha^1 \wedge \dots \wedge dx_\alpha^{m-1} = (dt + \sqrt{-1}d\theta) \wedge \Omega_D, \\ \omega_{\text{cyl}} &= \kappa_D + \frac{\sqrt{-1}}{2} \frac{dy \wedge d\bar{y}}{|y|^2} = \kappa_D + dt \wedge d\theta. \end{aligned}$$

Then there exist a holomorphic volume form Ω and an asymptotically cylindrical Ricci-flat Kähler form ω on $X = \overline{X} \setminus D$ such that

$$\begin{aligned} \Omega - \Omega_{\text{cyl}} &= d\zeta, \quad \omega - \omega_{\text{cyl}} = d\xi \quad \text{for some } \zeta \text{ and } \xi \text{ with} \\ \left| \nabla_{g_{\text{cyl}}}^j \zeta \right|_{g_{\text{cyl}}} &= O(e^{-\beta t}), \quad \left| \nabla_{g_{\text{cyl}}}^j \xi \right|_{g_{\text{cyl}}} = O(e^{-\beta t}) \quad \text{for all } j \geq 0 \text{ and } \beta \in (0, \min \{1/2, \sqrt{\lambda_1}\}), \end{aligned}$$

where λ_1 is the first eigenvalue of the Laplacian $\Delta_{g_D + d\theta^2}$ acting on $D \times S^1$ with g_D the metric associated with κ_D .

A pair (Ω, ω) consisting of a holomorphic volume form Ω and a Ricci-flat Kähler form ω on an m -dimensional Kähler manifold normalized so that

$$\frac{\omega^m}{m!} = \frac{(\sqrt{-1})^{m^2}}{2^m} \Omega \wedge \bar{\Omega} \quad (= \text{the volume form})$$

is called a *Calabi-Yau structure*. The above theorem states that there exists a Calabi-Yau structure (Ω, ω) on X asymptotic to a cylindrical Calabi-Yau structure $(\Omega_{\text{cyl}}, \omega_{\text{cyl}})$ on $N_{D/\bar{X}} \setminus D$ if we multiply Ω by some constant.

Theorem 3.8. *The statement in Theorem 3.7 also holds when (\bar{X}, D) is an orbifold admissible pair.*

Proof. This is essentially the same as the modification of the Calabi-Yau theorem for compact orbifolds. See [2], Chapter 3.6. \square

3.3. Kähler orbifolds with an antiholomorphic involution and $\text{Spin}(7)$ manifolds.

3.3.1. *Two basic examples of ALE $\text{Spin}(7)$ -manifolds.* Let Φ_0 be the standard $\text{Spin}(7)$ -structure on $\mathbb{R}^8 = \{(x_1, x_2, \dots, x_8)\}$. Let α, β act on \mathbb{R}^8 by

$$\begin{aligned} \alpha : (x_1, x_2, \dots, x_8) &\mapsto (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7), \\ \beta : (x_1, x_2, \dots, x_8) &\mapsto (x_3, -x_4, -x_1, x_2, x_7, -x_8, -x_5, x_6). \end{aligned}$$

Then α, β satisfy $\alpha^4 = \beta^4 = \text{id}_{\mathbb{R}^8}$, $\alpha\beta = \beta\alpha^3$ and $\alpha^*\Phi_0 = \beta^*\Phi_0 = \Phi_0$, so that the group $G = \langle \alpha, \beta \rangle$ is a subgroup of $\text{Spin}(7)$. Define complex coordinates (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) on \mathbb{R}^8 by

$$\begin{cases} z_1 = x_1 + \sqrt{-1}x_2 \\ z_2 = x_3 + \sqrt{-1}x_4 \\ z_3 = x_5 + \sqrt{-1}x_6 \\ z_4 = x_7 + \sqrt{-1}x_8, \end{cases} \quad \begin{cases} w_1 = -x_1 + \sqrt{-1}x_3 \\ w_2 = x_2 + \sqrt{-1}x_4 \\ w_3 = -x_5 + \sqrt{-1}x_7 \\ w_4 = x_6 + \sqrt{-1}x_8. \end{cases}$$

Then the coordinates (z_1, z_2, z_3, z_4) and (w_1, w_2, w_3, w_4) define Calabi-Yau structures (ω_0, Ω_0) and (ω'_0, Ω'_0) on \mathbb{R}^8 by

$$\begin{cases} \omega_0 = \frac{\sqrt{-1}}{2} \sum_{i=1}^4 dz_i \wedge d\bar{z}_i \\ \Omega_0 = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4, \end{cases} \quad \begin{cases} \omega'_0 = \frac{\sqrt{-1}}{2} \sum_{i=1}^4 dw_i \wedge d\bar{w}_i \\ \Omega'_0 = dw_1 \wedge dw_2 \wedge dw_3 \wedge dw_4, \end{cases}$$

both of which induce the $\text{Spin}(7)$ -structure Φ_0 by

$$\Phi_0 = \frac{1}{2} \omega_0 \wedge \omega_0 + \text{Re } \Omega_0 = \frac{1}{2} \omega'_0 \wedge \omega'_0 + \text{Re } \Omega'_0.$$

We see that α, β act on these coordinates as

$$\begin{aligned} \alpha : (z_1, z_2, z_3, z_4) &\mapsto (\sqrt{-1}z_1, \sqrt{-1}z_2, \sqrt{-1}z_3, \sqrt{-1}z_4) \\ \beta : (z_1, z_2, z_3, z_4) &\mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3), \\ \alpha : (w_1, w_2, w_3, w_4) &\mapsto (\bar{w}_2, -\bar{w}_1, \bar{w}_4, -\bar{w}_3) \\ \beta : (w_1, w_2, w_3, w_4) &\mapsto (\sqrt{-1}w_1, \sqrt{-1}w_2, \sqrt{-1}w_3, \sqrt{-1}w_4). \end{aligned}$$

Now we resolve the singularity of \mathbb{R}^8/G in two ways. Let us consider the action of α on \mathbb{C}^4 in the z -coordinates. Then we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{\beta} \curvearrowright \mathcal{Y}_1 & \longrightarrow & \mathcal{X}_1 \\ \text{crepant} \downarrow & & \downarrow \pi_1 \\ \underline{\beta} \curvearrowright \mathbb{C}^4 / \langle \alpha \rangle & \longrightarrow & \mathbb{R}^8 / G, \end{array}$$

where $\underline{\beta}$ is an antiholomorphic involution on $\mathbb{C}^4 / \langle \alpha \rangle$ induced by β , and $\tilde{\beta}$ is the lift of $\underline{\beta}$ which acts freely on \mathcal{Y}_1 . Since there exists an ALE Calabi-Yau structure $(\tilde{\omega}_1, \tilde{\Omega}_1)$ on \mathcal{Y}_1 with

$$\tilde{\beta}^* \tilde{\omega}_1 = -\tilde{\omega}_1, \quad \tilde{\beta}^* \tilde{\Omega}_1 = \overline{\tilde{\Omega}_1},$$

the induced torsion-free Spin(7)-structure $\tilde{\Phi}_1 = \frac{1}{2}\tilde{\omega}_1 \wedge \tilde{\omega}_1 + \text{Re } \tilde{\Omega}_1$ pushes down to a torsion-free Spin(7)-structure Φ_1 on \mathcal{X}_1 . This gives a resolution of \mathbb{R}^8/G by an ALE Spin(7)-manifold (\mathcal{X}_1, Φ_1) . Similarly, if we consider the action of β on \mathbb{C}^4 in the w -coordinate, then we have

$$\begin{array}{ccc} \tilde{\alpha} \curvearrowright \mathcal{Y}_2 & \longrightarrow & \mathcal{X}_2 \\ \text{crepant} \downarrow & & \downarrow \pi_2 \\ \underline{\alpha} \curvearrowright \mathbb{C}^4 / \langle \beta \rangle & \longrightarrow & \mathbb{R}^8 / G. \end{array}$$

If we consider

$$\begin{aligned} \phi : (z_1, z_2, z_3, z_4) &\longmapsto (w_1, w_2, w_3, w_4), & \text{that is,} \\ (x_1, x_2, \dots, x_8) &\longmapsto (-x_1, x_3, x_2, x_4, -x_5, x_7, x_6, x_8), \end{aligned}$$

then ϕ induces an isomorphism $\mathbb{C}^4 / \langle \alpha \rangle \xrightarrow{\cong} \mathbb{C}^4 / \langle \beta \rangle$, which lifts to an isomorphism $\tilde{\phi} : \mathcal{Y}_1 \xrightarrow{\cong} \mathcal{Y}_2$. Let Φ_2 be a Spin(7)-structure on \mathcal{X}_2 to which the Spin(7)-structure $(\tilde{\phi}^{-1})^* \tilde{\Phi}_1$ on \mathcal{Y}_2 pushes down. Then (\mathcal{X}_2, Φ_2) is also an ALE Spin(7)-manifold which resolves \mathbb{R}^8/G , but $\mathcal{X}_1, \mathcal{X}_2$ are *topologically distinct* because ϕ does not commute with α, β , so that the isomorphism ϕ acts nontrivially on \mathbb{R}^8/G .

Proposition 3.9 (Joyce [11], Section 15.1.1). *Let (\mathcal{X}_s, Φ_s) for $s = 1, 2$ be ALE Spin(7)-manifolds as above. Then the fundamental group of \mathcal{X}_s is \mathbb{Z}_2 , and*

$$(3.2) \quad b^i(\mathcal{X}_s) = \begin{cases} 1 & \text{if } i = 0, 4 \\ 0 & \text{otherwise,} \end{cases} \quad \text{so that } \chi(\mathcal{X}_s) = 2.$$

3.3.2. Compatible antiholomorphic involutions on orbifold admissible pairs.

Proposition 3.10. *Let X be a complex orbifold and $\sigma : X \rightarrow X$ be an antiholomorphic involution. Suppose S is a complex submanifold of X such that σ preserves and acts freely on S . Then σ lifts to a unique antiholomorphic involution $\tilde{\sigma}$ on the blow-up $\varpi : \text{Bl}_S(X) \dashrightarrow X$ of X along S such that $\tilde{\sigma}$ preserves and acts freely on $\varpi^{-1}(S)$.*

Proof. Let $m = \dim_{\mathbb{C}} X$ and $k = \dim_{\mathbb{C}} S$. Fix a point $x \in S$. It is enough to find a lift $\tilde{\sigma}$ of σ acting on a neighborhood of $\varpi^{-1}(x)$ in $\text{Bl}_S(X)$.

First we consider local coordinates near x and $\sigma(x)$ in X . We can choose a neighborhood U of $x \in S$ and local coordinates $(\mathbf{y}, \mathbf{z}) = (y_1, \dots, y_k, z_1, \dots, z_{m-k})$ on U such that $S \cap U = \{\mathbf{z} = \mathbf{0}\}$. We can similarly choose local coordinates $(\mathbf{y}', \mathbf{z}') = (y'_1, \dots, y'_k, z'_1, \dots, z'_{m-k})$ on $\sigma(U)$ such that $\sigma(S \cap U) = \{\mathbf{z}' = \mathbf{0}\}$ and

$$(\mathbf{y}', \mathbf{z}') = \sigma(\mathbf{y}, \mathbf{z}) = (\alpha(\mathbf{y}, \mathbf{z}), \beta(\mathbf{y}, \mathbf{z}))$$

for some antiholomorphic functions $\alpha : \mathbb{C}^m \rightarrow \mathbb{C}^k$ and $\beta : \mathbb{C}^m \rightarrow \mathbb{C}^{m-k}$. Also, $\sigma(S) = S$ yields that for $(\mathbf{y}, \mathbf{0}) \in S \cap U$

$$(3.3) \quad \sigma(\mathbf{y}, \mathbf{0}) = (\alpha(\mathbf{y}, \mathbf{0}), \mathbf{0}), \quad \text{that is, } \beta(\mathbf{y}, \mathbf{0}) = \mathbf{0}.$$

Since σ is an antiholomorphic diffeomorphism on X , the matrix $(\overline{\partial}_i \sigma_j(\mathbf{y}, \mathbf{z}))_{1 \leq i, j \leq m}$ is invertible for all $(\mathbf{y}, \mathbf{z}) \in U$, where $\overline{\partial}_i$ is the antiholomorphic partial differentiation with respect to the i -th variable. In particular, it follows from (3.3) that for $(\mathbf{y}, \mathbf{0}) \in S \cap U$ we have

$$(\overline{\partial}_i \sigma_j(\mathbf{y}, \mathbf{0}))_{1 \leq i, j \leq m} = \begin{pmatrix} (\overline{\partial}_i \alpha_j(\mathbf{y}, \mathbf{0}))_{1 \leq i, j \leq k} & O \\ (\overline{\partial}_{k+i} \alpha_j(\mathbf{y}, \mathbf{0}))_{1 \leq i \leq m-k, 1 \leq j \leq k} & (\overline{\partial}_{k+i} \beta_j(\mathbf{y}, \mathbf{0}))_{1 \leq i, j \leq m-k} \end{pmatrix},$$

so that $(\overline{\partial}_{k+i} \beta_j(\mathbf{y}, \mathbf{0}))_{1 \leq i, j \leq m-k}$ is also invertible. Thus we can expand $\beta(\mathbf{y}, \mathbf{z})$ for small \mathbf{z} as

$$(3.4) \quad \beta(\mathbf{y}, \mathbf{z}) = \sum_{i=1}^{m-k} \overline{\partial}_{k+i} \beta(\mathbf{y}, \mathbf{0}) \overline{z}_i + O(|\mathbf{z}|^2).$$

Next we consider local coordinates near $\varpi^{-1}(x)$ and $\varpi^{-1}(\sigma(x))$ in $\text{Bl}_S(X)$. Local coordinates of $\text{Bl}_S(X)$ on $\varpi^{-1}(U)$ are written as

$$\{ (\mathbf{y}, \mathbf{z}, [\zeta]) \in \mathbb{C}^m \times \mathbb{C}P^{m-k-1} \mid z_i \zeta_j = z_j \zeta_i \text{ for all } i, j \in \{1, \dots, m-k\} \},$$

where $\zeta = (\zeta_1, \dots, \zeta_{m-k}) \in \mathbb{C}^{m-k}$. Similarly, local coordinates of $\text{Bl}_S(X)$ on $\varpi^{-1}(\sigma(U))$ are written as

$$\{(\mathbf{y}', \mathbf{z}', [\zeta']) \in \mathbb{C}^m \times \mathbb{C}P^{m-k-1} \mid z'_i \zeta'_j = z'_j \zeta'_i \text{ for all } i, j \in \{1, \dots, m-k\}\}.$$

Thus we have

$$\begin{aligned} \varpi^{-1}(\mathbf{y}, \mathbf{z}) &= \{(\mathbf{y}, \mathbf{z}, [\mathbf{z}])\} && \text{for } (\mathbf{y}, \mathbf{z}) \in U \setminus S \text{ (and so } \mathbf{z} \neq \mathbf{0}), \\ \varpi^{-1}(\mathbf{y}, \mathbf{0}) &= \{(\mathbf{y}, \mathbf{0}, [\zeta]) \mid [\zeta] \in \mathbb{C}P^{m-k-1}\} && \text{for } (\mathbf{y}, \mathbf{0}) \in S \cap U. \end{aligned}$$

Now we shall find a lift $\tilde{\sigma}$ of σ acting on $\varpi^{-1}(U)$. For $(\mathbf{y}, \mathbf{z}) \in U \setminus S$, we must have

$$\tilde{\sigma}(\mathbf{y}, \mathbf{z}, [\mathbf{z}]) = (\sigma(\mathbf{y}, \mathbf{z}), [\beta(\mathbf{z})]).$$

Then $\tilde{\sigma}$ extends naturally to $\varpi^{-1}(S \cap U)$ by continuity as

$$\begin{aligned} \tilde{\sigma}(\mathbf{y}, \mathbf{0}, [\zeta]) &= \lim_{\lambda \rightarrow 0} \tilde{\sigma}(\mathbf{y}, \lambda \zeta, [\lambda \zeta]) \\ &= \lim_{\lambda \rightarrow 0} (\alpha(\mathbf{y}, \lambda \zeta), \beta(\mathbf{y}, \lambda \zeta), [\beta(\mathbf{y}, \lambda \zeta)]) \\ &= \left(\alpha(\mathbf{y}, \mathbf{0}), \mathbf{0}, \left[\sum_{i=1}^{m-k} \overline{D}_{k+i} \beta(\mathbf{y}, \mathbf{0}) \zeta_i \right] \right), \end{aligned}$$

where we used the expansion in (3.4) for the last equality. This gives the desired action of $\tilde{\sigma}$ on the neighborhood $\varpi^{-1}(U)$ of $\varpi^{-1}(x)$ in $\text{Bl}_S(X)$. \square

Definition 3.11. Let \overline{X} be a four-dimensional compact Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, such that (\overline{X}, D) is an orbifold admissible pair. An antiholomorphic involution σ on \overline{X} is said to be *compatible with* (\overline{X}, D) if the following conditions hold:

(f) We can choose a defining function w on a neighborhood U_D of D given in Lemma 3.4 so that

$$(3.5) \quad \sigma^* w = \overline{w},$$

where \overline{f} for a complex function f is defined by $\overline{f}(x) = \overline{f(x)}$.

(g) $(\overline{X})^\sigma = \text{Sing } \overline{X}$, where $(\overline{X})^\sigma$ is the fixed point set of the action of σ on \overline{X} .

Note that (3.5) in condition (f) implies $\sigma(D) = D$, and $\sigma_D = \sigma|_D$ yields an antiholomorphic involution on D .

Lemma 3.12. Let σ_{cyl} be an antiholomorphic involution on $N_{D/\overline{X}}$ defined by

$$(3.6) \quad \sigma_{\text{cyl}}(x_\alpha, y) = (\sigma_D(x_\alpha), \overline{y}) \quad \text{for } (x_\alpha, y) \in (U_\alpha \cap D) \times \mathbb{C} \subset N_{D/\overline{X}}.$$

Then we have

$$\sigma(z_\alpha, w) = \sigma_{\text{cyl}}(x_\alpha, y) + O(e^{-t}).$$

Proof. Using (3.5), we can write $\sigma(z_\alpha, w)$ as

$$(3.7) \quad \sigma(z_\alpha, w) = (\sigma_1(z_\alpha, w), \overline{w}) \quad \text{with } \sigma_1(x_\alpha, 0) = \sigma_D(x_\alpha).$$

Thus the assertion follows from Lemma 3.3 and (3.7). \square

Since the cylindrical parameter t is defined by $y = \exp(-t - \sqrt{-1}\theta)$, we have

$$\sigma_{\text{cyl}}^* t = t, \quad \sigma_{\text{cyl}}^* \theta = -\theta$$

and thus

$$(N_{D/\overline{X}} \setminus D) / \langle \sigma_{\text{cyl}} \rangle \simeq ((D \times S^1) / \langle \sigma_{D \times S^1, \text{cyl}} \rangle) \times \mathbb{R}_+,$$

where $\sigma_{D \times S^1, \text{cyl}}$ acts on $D \times S^1$ as

$$(3.8) \quad \sigma_{D \times S^1, \text{cyl}}(x_\alpha, \theta) = (\sigma_D(x_\alpha), -\theta).$$

One can prove the following result by Theorem 3.8 and an argument as used in the proof of [11], Proposition 15.2.2.

Theorem 3.13. *Let (\overline{X}, ω') be a four-dimensional Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, such that (\overline{X}, D) is an orbifold admissible pair with a compatible antiholomorphic involution σ . Then there exists an asymptotically cylindrical Calabi-Yau structure (ω, Ω) on $X = \overline{X} \setminus D$ asymptotic to $(\omega_{\text{cyl}}, \Omega_{\text{cyl}})$ on $N \setminus D$, such that*

$$\sigma^*g = g, \quad \sigma^*\omega = -\omega, \quad \sigma^*\Omega = \overline{\Omega},$$

where $N = N_{D/\overline{X}}$ and g is the Riemannian metric on X associated with (ω, Ω) . Thus the torsion-free $\text{Spin}(7)$ -structure $\frac{1}{2}\omega \wedge \omega + \text{Re } \Omega$ on X pushes down to a torsion-free $\text{Spin}(7)$ -structure Φ on $X/\langle\sigma\rangle$. Also, an antiholomorphic involution σ_{cyl} defined in (3.6) satisfies

$$\sigma_{\text{cyl}}^*g_{\text{cyl}} = g_{\text{cyl}}, \quad \sigma_{\text{cyl}}^*\omega_{\text{cyl}} = -\omega_{\text{cyl}}, \quad \sigma_{\text{cyl}}^*\Omega_{\text{cyl}} = \overline{\Omega_{\text{cyl}}},$$

so that the torsion-free $\text{Spin}(7)$ -structure $\frac{1}{2}\omega_{\text{cyl}} \wedge \omega_{\text{cyl}} + \text{Re } \Omega_{\text{cyl}}$ pushes down to a torsion-free $\text{Spin}(7)$ -structure Φ_{cyl} . We have

$$(3.9) \quad \begin{aligned} \Phi - \Phi_{\text{cyl}} &= d\Xi, \quad \text{for some } \Xi \text{ with} \\ \left| \nabla_{g_{\text{cyl}}}^j \Xi \right|_{g_{\text{cyl}}} &= O(e^{-\beta t}), \quad \text{for all } j \geq 0 \text{ and } \beta \in (0, \min\{1/2, \sqrt{\lambda_1}\}), \end{aligned}$$

where λ_1 is the constant given in Theorem 3.7. Hence $(X/\langle\sigma\rangle, \Phi)$ is an asymptotically cylindrical $\text{Spin}(7)$ -manifold, with the asymptotic model

$$\begin{aligned} (N \setminus D)/\langle\sigma_{\text{cyl}}\rangle &\simeq (D \times S^1)/\langle\sigma_{D \times S^1, \text{cyl}}\rangle \times \mathbb{R}_+ = \{([x_\alpha, \theta], t)\}, \\ \text{where } [x_\alpha, \theta] &= [\sigma_D(x_\alpha), -\theta] \quad \text{in } (D \times S^1)/\langle\sigma_{D \times S^1, \text{cyl}}\rangle. \end{aligned}$$

Theorem 3.14 (Joyce [11], Proposition 15.2.3 and Corollary 15.2.4). *All isolated singular points in $X/\langle\sigma\rangle$ are modelled on \mathbb{R}^8/G given in Section 3.3.1. For each $p \in \text{Sing } X/\langle\sigma\rangle$ there exists an isomorphism $\iota_p : \mathbb{R}^8/G \rightarrow T_p(X/\langle\sigma\rangle)$, which identifies the $\text{Spin}(7)$ -structures Φ_0 on \mathbb{R}^8 and Φ on $T_p(X/\langle\sigma\rangle)$.*

3.4. Gluing orbifold admissible pairs divided by compatible antiholomorphic involutions. In this subsection we will only consider admissible pairs (\overline{X}, D) with $\dim_{\mathbb{C}} \overline{X} = 4$. Also, we will denote $N = N_{D/\overline{X}}$ and $X = \overline{X} \setminus D$.

3.4.1. The gluing condition. Let (\overline{X}, ω') be a four-dimensional compact Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, and (\overline{X}, D) be an orbifold admissible pair with a compatible antiholomorphic involution σ . Then we obtained in Theorem 3.13 an asymptotically cylindrical, torsion-free $\text{Spin}(7)$ -manifold (X, Φ) , with the asymptotic model $(N \setminus D, \Phi_{\text{cyl}})$.

Next we consider the condition under which we can glue together $X_1/\langle\sigma_1\rangle$ and $X_2/\langle\sigma_2\rangle$ obtained from orbifold admissible pairs (\overline{X}_1, D_1) and (\overline{X}_2, D_2) with antiholomorphic involutions σ_i . For gluing $X_1/\langle\sigma_1\rangle$ and $X_2/\langle\sigma_2\rangle$ to obtain a manifold with a $\text{Spin}(7)$ -structure with small torsion, we would like $(X_1/\langle\sigma_1\rangle, \Phi_1)$ and $(X_2/\langle\sigma_2\rangle, \Phi_2)$ to have the same asymptotic model. Thus we put the following

Gluing condition: There exists an isomorphism $\tilde{f} : D_1 \rightarrow D_2$ between the cross-sections of the cylindrical ends of $\overline{X}_i \setminus D_i$ with

$$\tilde{f} \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ \tilde{f},$$

such that

$$(3.10) \quad \tilde{f}_T^* \left(\frac{1}{2} \omega_{2, \text{cyl}} \wedge \omega_{2, \text{cyl}} + \text{Re } \Omega_{2, \text{cyl}} \right) = \frac{1}{2} \omega_{1, \text{cyl}} \wedge \omega_{1, \text{cyl}} + \text{Re } \Omega_{1, \text{cyl}},$$

where $\tilde{f}_T : D_1 \times S^1 \times (0, 2T) \rightarrow D_2 \times S^1 \times (0, 2T)$ is defined by

$$\tilde{f}_T(x_1, \theta_1, t) = (\tilde{f}(x_1), -\theta_1, 2T - t) \quad \text{for } (x_1, \theta_1, t) \in D_1 \times S^1 \times (0, 2T).$$

Lemma 3.15. *If $\tilde{f} : D_1 \rightarrow D_2$ is an isomorphism satisfying $\tilde{f} \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ \tilde{f}$ and $\tilde{f}^* \kappa_{D_2} = \kappa_{D_1}$. Then the gluing condition (3.10) holds, where we change the sign of $\Omega_{2, \text{cyl}}$ (and also the sign of Ω_2 correspondingly).*

Proof. It follows by a straightforward calculation using (3.1) and Lemma 3.12. \square

The above \tilde{f} and \tilde{f}_T pushes down to maps

$$\begin{aligned} f : D_1 / \langle \sigma_{D_1} \rangle &\longrightarrow D_2 / \langle \sigma_{D_2} \rangle, \\ f_T : (D_1 \times S^1) / \langle \sigma_{D_1 \times S^1, \text{cyl}} \rangle \times (0, 2T) &\longrightarrow (D_2 \times S^1) / \langle \sigma_{D_2 \times S^1, \text{cyl}} \rangle \times (0, 2T), \\ \text{with } f([x_1]) &= ([\tilde{f}(x_1)]), \quad f_T([x_1, \theta_1], t) = ([\tilde{f}(x_1), -\theta_1], 2T - t) \end{aligned}$$

such that

$$f_T^* \Phi_{2, \text{cyl}} = \Phi_{1, \text{cyl}}.$$

3.4.2. $\text{Spin}(7)$ -structures with small torsion. Now we shall glue $X_1 / \langle \sigma_1 \rangle$ and $X_2 / \langle \sigma_2 \rangle$ under the gluing condition (3.10). Let $\rho : \mathbb{R} \longrightarrow [0, 1]$ denote a cut-off function

$$\rho(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ 0 & \text{if } x \geq 1, \end{cases}$$

and define $\rho_T : \mathbb{R} \longrightarrow [0, 1]$ by

$$\rho_T(x) = \rho(x - T + 1) = \begin{cases} 1 & \text{if } x \leq T - 1, \\ 0 & \text{if } x \geq T. \end{cases}$$

Setting an approximating Calabi-Yau structure $(\Omega_{i,T}, \omega_{i,T})$ on X_i by

$$\Omega_{i,T} = \begin{cases} \Omega_i - d(1 - \rho_{T-1})\zeta_i & \text{on } \{t_i \leq T - 1\}, \\ \Omega_{i, \text{cyl}} + d\rho_{T-1}\zeta_i & \text{on } \{t_i \geq T - 2\} \end{cases}$$

and similarly

$$\omega_{i,T} = \begin{cases} \omega_i - d(1 - \rho_{T-1})\xi_i & \text{on } \{t_i \leq T - 1\}, \\ \omega_{i, \text{cyl}} + d\rho_{T-1}\xi_i & \text{on } \{t_i \geq T - 2\}, \end{cases}$$

we can define a d-closed 4-form $\tilde{\Phi}_{i,T}$ on each $X_i / \langle \sigma_i \rangle$ by

$$\tilde{\Phi}_{i,T} = \pi_{i*} \left(\frac{1}{2} \omega_{i,T} \wedge \omega_{i,T} + \text{Re } \Omega_T \right),$$

where $\pi_i : X_i \longrightarrow X_i / \langle \sigma_i \rangle$ are projections. We see that $\tilde{\Phi}_{i,T}$ satisfies

$$\tilde{\Phi}_{i,T} = \begin{cases} \Phi_i & \text{on } \{t_i < T - 2\}, \\ \Phi_{i, \text{cyl}} & \text{on } \{t_i > T - 1\} \end{cases}$$

and from (3.9) that

$$(3.11) \quad \left| \tilde{\Phi}_{i,T} - \Phi_{i, \text{cyl}} \right|_{g_{\Phi_{i, \text{cyl}}}} = O(e^{-\beta T}) \quad \text{for all } \beta \in (0, \min \{ 1/2, \sqrt{\lambda_1} \}).$$

Let $X_{1,T} = \{t_1 < T + 1\} \subset X_1$ and $X_{2,T} = \{t_2 < T + 1\} \subset X_2$. We glue $X_{1,T} / \langle \sigma_1 \rangle$ and $X_{2,T} / \langle \sigma_2 \rangle$ along $((D_1 \times S^1) / \langle \sigma_{D_1 \times S^1, \text{cyl}} \rangle) \times \{T - 1 < t_1 < T + 1\} \subset X_{1,T} / \langle \sigma_1 \rangle$ and $((D_2 \times S^1) / \langle \sigma_{D_2 \times S^1, \text{cyl}} \rangle) \times \{T - 1 < t_2 < T + 1\} \subset X_{2,T} / \langle \sigma_2 \rangle$ to construct a compact 8-orbifold using the gluing map f_T (more precisely, $F_T = \varphi_2 \circ f_T \circ \varphi_1^{-1}$, where φ_1 and φ_2 are the diffeomorphisms given in Lemma 3.3). We denote this orbifold by M_T^∇ (the upper index ∇ indicates singularities to be resolved). Also, we can glue together $\tilde{\Phi}_{1,T}$ and $\tilde{\Phi}_{2,T}$ to obtain a d-closed 4-form $\tilde{\Phi}_T$ on M_T^∇ by Lemma 3.15. There exists a positive constant T_* such that $\tilde{\Phi}_T \in C^\infty(\mathcal{T}(M_T^\nabla))$ for any $T > T_*$. This $\tilde{\Phi}_T$ is what was discussed right after Theorem 2.7, from which we can define a $\text{Spin}(7)$ -structure Φ_T with small torsion by $\Phi_T = \Theta(\tilde{\Phi}_T)$. Letting $\phi_T = \tilde{\Phi}_T - \Phi_T$, we have $d\phi_T + d\Phi_T = 0$.

Proposition 3.16. *Let $T > T_*$. Then there exist constants $A_{p,k,\beta}$ independent of T such that for $\beta \in (0, \min \{ 1/2, \sqrt{\lambda_1} \})$ we have*

$$\|\phi_T\|_{L_k^p} \leq A_{p,k,\beta} e^{-\beta T},$$

where all norms are measured using g_{Φ_T} .

Proof. These estimates follow in a straightforward way from Theorem 3.7 and (3.11) by an argument similar to those in [5], Section 3.5. \square

3.4.3. *Resolving M_T^∇ by ALE Spin(7)-manifolds \mathcal{X}_1 and \mathcal{X}_2 .* The material in this section is taken from [11] Section 15.2.2. Let $p \in \text{Sing } M_T^\nabla$ and $\iota_p : \mathbb{R}^8/G \rightarrow T_p M_T^\nabla$ as in Theorem 3.14. Let $\exp_p : T_p M_T^\nabla \rightarrow M_T^\nabla$ be the exponential map. Then $\psi_p = \exp_p \circ \iota_p$ maps each ball $B_{2\zeta}(\mathbb{R}^8/G)$ of 2ζ in \mathbb{R}^8/G to a neighborhood of $p \in M_T^\nabla$. Choose $\zeta > 0$ small so that $U_p = \exp_p \circ \iota_p(B_{2\zeta}(\mathbb{R}^8/G))$ satisfy $U_p \cap U_{p'} = \emptyset$ and $U_p \cap \{t_i > T - 2\} = \emptyset$ for any $p, p' \in M_T^\nabla$ with $p \neq p'$ and for any $T > T_*$.

Proposition 3.17 (Joyce [11], Proposition 15.2.6). *There exist a smooth 3-form σ_p on $B_{2\zeta}(\mathbb{R}^8/G)$ for each $p \in \text{Sing } M_T^\nabla$ and a constant $C_1 > 0$ independent of $T > T_*$, such that*

$$\psi_p^* \Phi_T^\nabla - \Phi_0 = d\sigma_p, \quad |\nabla^\ell \sigma_p| \leq C_1 r^{3-\ell} \quad \text{for } \ell = 0, 1, 2$$

on $B_{2\zeta}(\mathbb{R}^8/G)$. Here $|\cdot|$ and ∇ is defined by the metric g_0 induced by Φ_0 , and r is the radius function on \mathbb{R}^8/G .

Let $\pi_s : \mathcal{X}_s \rightarrow \mathbb{R}^8/G$ be the projections given in Section 3.3.1. For each $\epsilon \in (0, 1]$ and $s = 1, 2$ let $\mathcal{X}_s^\epsilon = \mathcal{X}_s$, define a Spin(7)-structure $\Phi_s^\epsilon = \epsilon^4 \Phi_s$ and define $\pi_s^\epsilon : \mathcal{X}_s^\epsilon \rightarrow \mathbb{R}^8/G$ by $\pi_s^\epsilon = \epsilon \pi_s$. Then $(\mathcal{X}_s^\epsilon, \Phi_s^\epsilon)$ is an ALE Spin(7)-manifold asymptotic to \mathbb{R}^8/G .

Proposition 3.18 (Joyce [11], equation (15.6)). *There exist a constant $C_2 > 0$ independent of $T > T_*$, and a smooth 3-form τ_s^ϵ on $(\mathbb{R}^8/G) \setminus B_{\epsilon\zeta}(\mathbb{R}^8/G)$ such that*

$$(\pi_s^\epsilon)^* \Phi_s^\epsilon - \Phi_0 = d\tau_s^\epsilon, \quad |\nabla^\ell \tau_s^\epsilon| \leq C_2 \epsilon^8 r^{-7-\ell} \quad \text{for } \ell = 0, 1, 2$$

on $(\mathbb{R}^8/G) \setminus B_{\epsilon\zeta}(\mathbb{R}^8/G)$, where $|\cdot|$ and ∇ is defined using the metric g_0 induced by Φ_0 .

Now we glue together

$$U_T^\epsilon = M_T^\nabla \setminus \bigcup_{p \in \text{Sing } M_T^\nabla} \psi_p(\overline{B_{\epsilon^{4/5}\zeta}}(\mathbb{R}^8/G)) \quad \text{and} \quad V_p^\epsilon = (\pi_{s_p}^\epsilon)^{-1}(B_{2\epsilon^{4/5}\zeta}(\mathbb{R}^8/G)), \quad s_p \in \{1, 2\},$$

along the regions diffeomorphic to

$$B_{2\epsilon^{4/5}\zeta}(\mathbb{R}^8/G) \setminus \overline{B_{\epsilon^{4/5}\zeta}}(\mathbb{R}^8/G) \quad \text{in } \mathbb{R}^8/G,$$

to obtain a compact 8-manifold M_T^ϵ . Let η be a smooth cut-off function with $\eta(x) = 0$ for $x \leq \zeta$ and $\eta(x) = 1$ for $x \geq 2\zeta$. Choosing $s_p \in \{1, 2\}$ for each $p \in \text{Sing } M_T^\nabla$, we can also glue the Spin(7)-structures Φ_T^∇ on M_T^∇ and $\Phi_{s_p}^\epsilon$ on $\mathcal{X}_{s_p}^\epsilon$ to obtain a closed 4-form $\tilde{\Phi}_T^\epsilon$ on M_T^ϵ by

$$\tilde{\Phi}_T^\epsilon = \Phi_0 + d\left(\eta(\epsilon^{-4/5}r)\sigma_p\right) + d\left((1 - \eta(\epsilon^{-4/5}r))\tau_{s_p}^\epsilon\right) \quad \text{on } U_T^\epsilon \cap V_p^\epsilon.$$

Now we set $\epsilon = \exp(-\gamma T)$ for some constant $\gamma > 0$ to be determined later, and define $M^\epsilon = M_T^\epsilon$, $\tilde{\Phi}^\epsilon = \tilde{\Phi}_T^\epsilon$ and $U^\epsilon = U_T^\epsilon$.

Proposition 3.19 (Joyce [11], Proposition 15.2.9). *If $s_p = 1$ for all $p \in \text{Sing } M_T^\nabla$, then the fundamental group of M^ϵ is \mathbb{Z}_2 . Otherwise, M^ϵ is simply-connected.*

Lemma 3.20 (Joyce [11], Lemma 15.2.11). *There exists a constant $C_3 > 0$ independent of $T > T_*$ such that $\tilde{\Phi}_T^\epsilon$ satisfies*

$$|\tilde{\Phi}^\epsilon - \Phi_0| \leq C_3 \epsilon^{8/5}, \quad |\nabla(\tilde{\Phi}^\epsilon - \Phi_0)| \leq \epsilon^{4/5}$$

on $U^\epsilon \cap V_p^\epsilon$ for all $p \in \text{Sing } M_T^\nabla$, where $|\cdot|$ and ∇ is defined using the metric g_0 induced by Φ_0 .

Letting $\Phi^\epsilon = \Theta(\tilde{\Phi}^\epsilon)$ and $\phi^\epsilon = \tilde{\Phi}^\epsilon - \Phi^\epsilon$, we have $d\phi^\epsilon + d\Phi^\epsilon = 0$.

Theorem 3.21. *There exist a family $(M^\epsilon, \Phi^\epsilon)$ of smooth 8-manifolds with a Spin(7)-structure with small torsion and resolutions $\pi^\epsilon : M^\epsilon \rightarrow M^\nabla$ for $\epsilon \in (0, 1]$ such that we have*

- (i) $\|\phi^\epsilon\|_{L^2} \leq \lambda \epsilon^{24/5}$ and $\|d\phi^\epsilon\|_{L^{10}} \leq \lambda \epsilon^{36/25}$,
- (ii) the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \mu \epsilon$, and
- (iii) the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq \nu \epsilon^{-2}$,

where all norms are measured using the metric g^ϵ on M^ϵ induced by Φ^ϵ .

Proof. The proof is almost the same as that of [11], Proposition 15.2.13 except for the contributions from the cylinder, which is diffeomorphic to $\Sigma \times (0, 2T)$ with $\Sigma = (D \times S^1)/\langle \sigma_{D \times S^1, \text{cyl}} \rangle$. Joyce proved the following estimates using Lemma 3.20:

$$\sum_{p \in \text{Sing } M_T^\nabla} \int_{U^\epsilon \cap V_p^\epsilon} |\phi^\epsilon|^2 \leq \lambda^2 \epsilon^{48/5}, \quad \sum_{p \in M_T^\nabla} \int_{U^\epsilon \cap V_p^\epsilon} |d\phi^\epsilon|^2 \leq \lambda^{10} \epsilon^{72/5},$$

Meanwhile, Proposition 3.16 gives

$$\int_{\Sigma \times (0, 2T)} |\phi_T|^2 \leq 2A_\beta^2 e^{-2\beta T}, \quad \int_{\Sigma \times (0, 2T)} |d\phi_T|^{10} \leq 2A_\beta^{10} e^{-10\beta T},$$

where we take $\beta \in (0, \max\{1/2, \sqrt{\lambda_1}\})$ and $A_\beta = \max\{A_{2,0,\beta}, A_{10,1,\beta}\}$. Now if we choose $\gamma > 0$ for $\epsilon = e^{-\gamma T}$ so that $\frac{24}{5}\gamma \leq \beta$, then we have $e^{-2\beta T} \leq \epsilon^{48/5}$ and $e^{-10\beta T} \leq \epsilon^{72/5}$. Summing up the above contributions and redefining λ to be $\max\{(\lambda^2 + 2A_\beta^2)^{1/2}, (\lambda^{10} + 2A_\beta^{10})^{1/10}\}$, we see that condition (i) holds. Conditions (ii) and (iii) are obvious. \square

3.5. Gluing theorems. First we give a gluing and a doubling construction of Calabi-Yau fourfolds from orbifold admissible pairs, which are generalizations of Theorem 3.10 and Corollary 3.11 in [7].

Theorem 3.22. *Let $(\overline{X}_1, \omega'_1)$ and $(\overline{X}_2, \omega'_2)$ be compact Kähler orbifolds with $\dim_{\mathbb{C}} \overline{X}_i = 4$ such that (\overline{X}_1, D_1) and (\overline{X}_2, D_2) are orbifold admissible pairs. Suppose there exists an isomorphism $f : D_1 \rightarrow D_2$ such that $f^* \kappa_2 = \kappa_1$, where κ_i is the unique Ricci-flat Kähler form on D_i in the Kähler class $[\omega'_i|_{D_i}]$. Then we can glue together the crepant resolutions of X_1 and X_2 along their cylindrical ends to obtain a compact simply-connected 8-manifold M . The manifold M admits a Riemannian metric with holonomy contained in $\text{Spin}(7)$. Moreover, if $\hat{A}(M) = 2$, then M is a Calabi-Yau fourfold, i.e., M admits a Ricci-flat Kähler metric with holonomy $\text{SU}(4)$.*

Corollary 3.23. *Let (\overline{X}, D) be an orbifold admissible pair with $\dim_{\mathbb{C}} \overline{X} = 4$. Then we can glue two copies of the crepant resolution of X along their cylindrical ends to obtain a compact simply-connected 8-manifold M . Then M admits a Riemannian metric with holonomy contained in $\text{Spin}(7)$. If $\hat{A}(M) = 2$, then the manifold M is a Calabi-Yau fourfold.*

Next we give a gluing and a doubling construction of compact $\text{Spin}(7)$ -manifolds.

Theorem 3.24. *Let $(\overline{X}_1, \omega'_1)$ and $(\overline{X}_2, \omega'_2)$ be four-dimensional compact Kähler orbifolds with singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, such that (\overline{X}_i, D_i) are orbifold admissible pairs with a compatible antiholomorphic involution σ_i . Suppose there exists an isomorphism $f : D_1 \rightarrow D_2$ such that $f \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ f$ and $f^* \kappa_2 = \kappa_1$, where κ_i is the unique Ricci-flat Kähler form in the Kähler class $[\omega'_i|_{D_i}]$. Then we can glue together $X_1/\langle \sigma_1 \rangle$ and $X_2/\langle \sigma_2 \rangle$ along their cylindrical ends to obtain a compact 8-orbifold M^∇ . There exists a compact simply-connected 8-manifold M which resolves M^∇ at $(\# \text{Sing } \overline{X}_1 + \# \text{Sing } \overline{X}_2)$ isolated singular points such that M admits a Riemannian metric with holonomy contained in $\text{Spin}(7)$. Furthermore if $\hat{A}(M) = 1$, then M is a compact $\text{Spin}(7)$ -manifold.*

Corollary 3.25. *Let (\overline{X}, ω') be a four-dimensional Kähler orbifold with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$, such that (\overline{X}, D) be an orbifold admissible pair with a compatible antiholomorphic involution σ . Then we can glue together two copies of $X/\langle \sigma \rangle = (\overline{X} \setminus D)/\langle \sigma \rangle$ to obtain a compact 8-orbifold M^∇ . There exists a compact simply-connected 8-manifold M which resolves M^∇ at $2(\# \text{Sing } \overline{X})$ isolated singular points such that M admits a Riemannian metric with holonomy contained in $\text{Spin}(7)$. Furthermore if $\hat{A}(M) = 1$, then M is a compact $\text{Spin}(7)$ -manifold.*

Proof of Theorem 3.24. By Proposition 3.19, there exists a choice $\{s_p \in \{1, 2\} \mid p \in \text{Sing } M^\nabla\}$ of resolutions by \mathcal{X}_{s_p} such that $M = M^\epsilon$ is simply-connected. The assertion for $\hat{A}(M) = 1$ in Theorem 3.24 follows directly from Theorem 2.8. Thus it remains to prove the existence of a torsion-free $\text{Spin}(7)$ -structure on M^ϵ for sufficiently small $\epsilon \in (0, 1]$. This is a consequence of the following

Theorem 3.26 (Joyce [11], Theorem 13.6.1). *Let λ, μ, ν be positive constants. Then there exists a positive constant ϵ_* such that whenever $0 < \epsilon < \epsilon_*$, the following is true.*

Let M be a compact 8-manifold and Φ a $\text{Spin}(7)$ -structure on M . Suppose ϕ is a smooth 4-form on M with $d\Phi + d\phi = 0$, and

- (1) $\|\phi\|_{L^2} \leq \lambda\epsilon^{13/3}$ and $\|d\phi\|_{L^{10}} \leq \lambda\epsilon^{7/5}$,
- (2) the injectivity radius $\delta(g)$ satisfies $\delta(g) \geq \mu\epsilon$, and
- (3) the Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq \nu\epsilon^{-2}$.

Let ϵ_1 be as in Lemma 2.6. Then there exists $\eta \in C^\infty(\wedge^4 T^*M)$ with $\|\eta\|_{C^0} < \epsilon_1$ such that $d\Theta(\Phi + \eta) = 0$. Hence the manifold M admits a torsion-free $\text{Spin}(7)$ -structure $\Theta(\Phi + \eta)$.

If we set $\phi = \phi^\epsilon$, then M^ϵ and ϕ^ϵ satisfy conditions (i)–(iii) in Theorem 3.21. Thus we can apply Theorem 3.26 to prove that Φ^ϵ can be deformed into a torsion-free $\text{Spin}(7)$ -structure for sufficiently small $\epsilon \in (0, 1]$. This completes the proof of Theorem 3.24. \square

4. NEW EXAMPLES OF COMPACT $\text{Spin}(7)$ -MANIFOLDS

In this and the following section, we give three examples of compact simply-connected 8-manifolds M obtained by Corollary 3.25 and Theorem 4.9. We will see that the \hat{A} -genus of M is 1 in each case. This shows that M is a compact $\text{Spin}(7)$ -manifold.

We shall calculate the Euler characteristics, Betti numbers and the signatures of these compact $\text{Spin}(7)$ -manifolds. The Euler characteristics of the resulting compact $\text{Spin}(7)$ -manifolds are 912, 1296 and 1680. Of these compact $\text{Spin}(7)$ -manifolds, we see that at least one with $\chi(M) = 1680$ is a new example.

4.1. Preliminaries. In order to find orbifold admissible pairs with a compatible antiholomorphic involution in Definitions 3.6 and 3.11 we will use some algebro-geometrical approach. In particular, hypersurfaces and complete intersections in weighted projective spaces are well-studied in the context of mirror symmetry for Calabi-Yau manifolds. First we will review some basics on weighted projective spaces. Next we will also explain notation on complete intersections in weighted projective spaces. See [8] for more details.

4.1.1. Basics on projective spaces. First we will observe the structure of the weighted projective space as a complex orbifold. Let a_0, \dots, a_n be positive integers with $\gcd(a_0, \dots, a_n) = 1$. Recall that the *weighted projective space* $\mathbb{C}P^n(a_0, \dots, a_n)$ is the quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where \mathbb{C}^* acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$\mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}^{n+1} \setminus \{0\}, \quad (w_0, \dots, w_n) \longmapsto (t^{a_0}w_0, \dots, t^{a_n}w_n)$$

for $t \in \mathbb{C}^*$. Let us fix the point $p = [1, 0, \dots, 0]$ in $\mathbb{C}P^n(a_0, \dots, a_n)$. Denote the stabilizer of p in \mathbb{C}^* by $(\mathbb{C}^*)_p$. Then the point $(1, 0, \dots, 0)$ in $\mathbb{C}^{n+1} \setminus \{0\}$ is taken to $(t^{a_0}, 0, \dots, 0)$ under the action of $t \in \mathbb{C}^*$. Thus we have an isomorphism

$$(\mathbb{C}^*)_p = \{t \in \mathbb{C}^* \mid t^{a_0} = 1\} \cong \mathbb{Z}_{a_0},$$

where \mathbb{Z}_{a_0} is a finite cyclic group of order a_0 . Let $[z_0, \dots, z_n]$ be the weighted homogeneous coordinates on $\mathbb{C}P^n(a_0, \dots, a_n)$. Then the affine open chart

$$U_0 = \{[z_0, \dots, z_n] \in \mathbb{C}P^n(a_0, \dots, a_n) \mid z_0 \neq 0\}$$

is isomorphic to $\mathbb{C}^n/\mathbb{Z}_{a_0}$ as follows. Taking an orbifold chart \tilde{U}_0 on $\mathbb{C}P^n(a_0, \dots, a_n)$ with a continuous map $\varphi : \tilde{U}_0 \longrightarrow U_0$, we consider affine coordinates (x_1, \dots, x_n) on \tilde{U}_0 with $x_j = z_j/z_0$. Then the stabilizer acts via

$$(4.1) \quad (x_1, \dots, x_n) \longmapsto (\zeta^{a_1}x_1, \dots, \zeta^{a_n}x_n),$$

where $\zeta \in (\mathbb{C}^*)_p$ is a primitive a_0 -th root of unity. This implies our desired result. Furthermore $p \in \mathbb{C}P^n(a_0, \dots, a_n)$ is a quotient singular point with a finite cyclic group \mathbb{Z}_{a_0} which acts on \mathbb{C}^n by (4.1). In particular, all singularities of $\mathbb{C}P^n(a_0, \dots, a_n)$ are cyclic quotient singularities.

Next we shall define $\mathbb{C}P^n(a_0, \dots, a_n)$ as a projective variety. Let R be the graded ring $\mathbb{C}[z_0, \dots, z_n]$. Suppose each variable z_i has the weight a_i . Then R has a natural weight decomposition $R = \bigoplus_{d=0}^{\infty} R_d$ where

R_d denotes the vector space spanned by all monomials $z_0^{d_0} \dots z_n^{d_n}$ with $\sum a_i d_i = d$. Elements of R_d are said to be *weighted homogeneous polynomials* of degree d and then $\mathbb{C}P^n(a_0, \dots, a_n)$ is defined by

$$\mathbb{C}P^n(a_0, \dots, a_n) = \text{Proj}(R).$$

For a given finitely generated graded ring R , $\text{Proj}(R)$ denotes the projective scheme which is an algebraic variety constructed by gluing affine varieties. Furthermore, if positive integers a_1, \dots, a_n have a common divisor, we have an isomorphism

$$\mathbb{C}P^n(a_0, \dots, a_n) \cong \mathbb{C}P^n(a_0, a_1/q, \dots, a_n/q)$$

where $q = \gcd(a_1, \dots, a_n)$. This yields the following property (see [8], Corollary. 5.9):

Proposition 4.1. *Let a_0, \dots, a_n be positive integers with $\gcd(a_0, \dots, a_n) = 1$. Then we have an isomorphism as varieties*

$$\mathbb{C}P^n(a_0, \dots, a_n) \cong \mathbb{C}P^n(b_0, \dots, b_n)$$

for some positive integers b_0, \dots, b_n with $\gcd(b_0, \dots, \widehat{b}_i, \dots, b_n) = 1$ for each i . Here the symbol \widehat{b}_i means that the entry b_i is omitted.

Hence it is natural to define the following.

Definition 4.2. A weighted projective space $\mathbb{C}P^n(a_0, \dots, a_n)$ is said to be *well-formed* if and only if $\gcd(a_0, \dots, \widehat{a}_i, \dots, a_n) = 1$ for each i .

Recall that the graded ring $R = \mathbb{C}[z_0, \dots, z_n]$ is given by $\deg z_i = a_i \in \mathbb{Z}_{>0}$. Let $S = \mathbb{C}[w_0, \dots, w_n]$ be the standard polynomial ring with $\deg w_i = 1$. Then we have the injective ring homomorphism

$$R \longrightarrow S, \quad z_i \longmapsto w_i^{a_i}.$$

This injective ring homomorphism induces the well-defined surjective morphism of varieties

$$(4.2) \quad \begin{aligned} \pi : \quad \text{Proj}(S) = \mathbb{C}P^n &\longrightarrow \text{Proj}(R) = \mathbb{C}P^n(a_0, \dots, a_n), \\ [w_0, \dots, w_n] &\longmapsto [z_0, \dots, z_n] = [w_0^{a_0}, \dots, w_n^{a_n}]. \end{aligned}$$

By abuse of notation, we *also* denote by π the canonical projection from $\mathbb{C}^{n+1} \setminus \{0\}$ onto $\mathbb{C}P^n(a_0, \dots, a_n)$:

$$\pi : \quad \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}P^n(a_0, \dots, a_n), \quad (w_0, \dots, w_n) \longmapsto [w_0^{a_0}, \dots, w_n^{a_n}].$$

For this canonical projection π and a subvariety $X \subset \mathbb{C}P^n(a_0, \dots, a_n)$, we define the *affine cone* C_X over X to be

$$C_X = \pi^{-1}(X) \cup \{0\} \quad \text{in} \quad \mathbb{C}^{n+1}.$$

Definition 4.3. A subvariety X of $\mathbb{C}P^n(a_0, \dots, a_n)$ is called *quasismooth* if C_X is smooth except at the origin.

Definition 4.4. Let X be a subvariety of $\mathbb{C}P^n(a_0, \dots, a_n)$ with codimension k . Then X is said to be *well-formed* if $\mathbb{C}P^n(a_0, \dots, a_n)$ is well-formed and X does not contain a codimension $k+1$ singular locus of $\mathbb{C}P^n(a_0, \dots, a_n)$.

4.1.2. Weighted complete intersections.

Definition 4.5. Let a_0, \dots, a_n be positive integers with $\gcd(a_0, \dots, a_n) = 1$ and $R = \mathbb{C}[z_0, \dots, z_n]$ be the graded ring with $\deg z_i = a_i$ as usual. Let f_1, \dots, f_k with $k \leq n+1$ be weighted homogeneous polynomials of the graded ring R with $\deg f_i = d_i$. Then $I = \langle f_1, \dots, f_k \rangle$ is a homogeneous ideal of R . We define X_I by

$$X_I = \text{Proj}(R/I) \subset \mathbb{C}P^n(a_0, \dots, a_n).$$

Then X_I is a *weighted complete intersection* of multidegree (d_1, \dots, d_k) if the defining ideal I can be generated by a regular sequence f_1, \dots, f_k . Here a sequence of elements f_1, \dots, f_k with $k \leq n+1$ in R is said to be a *regular sequence* if f_1 is not a zero-divisor in R and the class $[f_i]$ is not a zero-divisor in $R/\langle f_1, \dots, f_{i-1} \rangle$ for each $2 \leq i \leq k$.

Now we will state the following results which will be needed for our argument later on.

Lemma 4.6 (Fletcher [8], Lemma. 7.1). *Let $X \subset \mathbb{C}P^n(a_0, \dots, a_n)$ be a well-formed quasismooth weighted complete intersection with the defining ideal $I(X) = \langle f_1, \dots, f_k \rangle$. Suppose $\deg f_i = d_i$. Let A be the residue ring*

$$A = \frac{\mathbb{C}[z_0, \dots, z_n]}{\langle f_1, \dots, f_k \rangle}.$$

Since each f_i is homogeneous, the ring A decomposes into each graded piece: $A = \bigoplus_m A_m$. Then

$$H^q(X, \mathcal{O}_X(m)) \cong \begin{cases} A_m & \text{if } q = 0 \\ 0 & \text{if } q = 1, \dots, \dim_{\mathbb{C}} X - 1 \\ A_{\alpha-m} & \text{if } q = \dim_{\mathbb{C}} X \end{cases}$$

for all $m \in \mathbb{Z}$, where $\alpha = \sum_{\lambda=1}^k d_{\lambda} - \sum_{i=0}^n a_i$.

In particular, we have the following beautiful result for hypersurfaces.

Theorem 4.7 (Fletcher [8], Theorem 7.2). *Let f be the defining polynomial of a weighted hypersurface X in $\mathbb{C}P^n(a_0, \dots, a_n)$ with $\deg f = d$. The Jacobian ring $R(f)$ of f is the quotient ring*

$$R(f) = \frac{\mathbb{C}[z_0, \dots, z_n]}{\langle \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \rangle}.$$

Let $R(f)_m$ denote the m -th graded part of $R(f)$. Then the Hodge numbers of X are given by

$$h^{p,q}(X) = \begin{cases} 0 & \text{if } p+q \neq n-1, p \neq q \\ 1 & \text{if } p+q \neq n-1, p = q \\ \dim_{\mathbb{C}} R(f)_{qd+\alpha} & \text{if } p+q = n-1, p \neq q \\ \dim_{\mathbb{C}} R(f)_{qd+\alpha} + 1 & \text{if } p+q = n-1, p = q, \end{cases}$$

where $\alpha = d - \sum_{i=0}^n a_i$.

4.2. Orbifold admissible pairs with a compatible antiholomorphic involution from weighted complete intersections. Here we consider a situation where the gluing condition holds naturally. We first recall the following result, which provides a way of obtaining orbifold admissible pairs of *Fano type*.

Theorem 4.8 (Kovalev [12]). *Let V be a Fano four-orbifold with isolated singular points which have local crepant resolutions, $D \in |-K_V|$ a smooth Calabi-Yau divisor, and S a smooth surface in D representing the self-intersection class of $D \cdot D$ on V . Let $\varpi : \overline{X} \dashrightarrow V$ be the blow-up of V along the surface S . If we take the proper transform D' of D under the blow-up ϖ , then (\overline{X}, D') is an orbifold admissible pair. Moreover, $\varpi|_{D'}$ yields an isomorphism between D' and D , and so we may denote D' by D .*

Proof. See [12], Proposition 6.42 and Corollary 6.43. These results for Fano threefolds also hold for Fano four-orbifolds V . \square

The above orbifold admissible pair (\overline{X}, D) obtained from V and D is said to be of *Fano type*.

Next we consider a well-formed weighted projective space $W := \mathbb{C}P^{k+3}(a_0, a_1, \dots, a_{k+3})$ with $k \geq 1$. Let f_1, \dots, f_{k+1} be a regular sequence of weighted homogeneous polynomials such that

- (1) $\sum_{\lambda=1}^k d_{\lambda} = \sum_{i=0}^{k+3} a_i$, where $d_{\lambda} = \deg f_{\lambda}$,
- (2) V is a complete intersection defined by the ideal $I_{k-1} = \langle f_1, \dots, f_{k-1} \rangle$, with isolated singular points modelled on $\mathbb{C}^4/\mathbb{Z}_4$ (we set $I_0 = 0$ and $V = W$ when $k = 1$),
- (3) D is a smooth complete intersection defined by the ideal $I_k = \langle f_1, \dots, f_k \rangle$, so that $D \cap \text{Sing } V = \emptyset$, and
- (4) S is a smooth complete intersection defined by the ideal $I_{k+1} = \langle f_1, \dots, f_{k+1} \rangle$ with $\deg f_{k+1} = \deg f_k$.

Then V is a four-dimensional Fano orbifold with D a smooth anticanonical Calabi-Yau divisor, and S is a smooth surface in D representing $D \cdot D$ on V . Suppose there exists an antiholomorphic involution σ on W such that

- (5) $\sigma^* f_i = \overline{f_i}$ for $i = 1, \dots, k+1$ and σ acts freely on D and S , and
- (6) $V^{\sigma} = \text{Sing } V$, where $V^{\sigma} = \{x \in V \mid \sigma(x) = x\}$.

Then by Proposition 3.10, σ lifts to an antiholomorphic involution $\tilde{\sigma}$ on the blow-up $\varpi : \overline{X} := \text{Bl}_S(V) \dashrightarrow V$ such that $\tilde{\sigma}$ preserves and acts freely on the exceptional divisor $E := \varpi^{-1}(S)$. Let $[\mathbf{z}] = [z_0, \dots, z_4]$ be weighted homogeneous coordinates on W , with $\deg z_i = a_i$ for $i = 0, \dots, k+3$. We can describe the blow-up \overline{X} of V , the exceptional divisor E and the proper transform D' of D as

$$\begin{aligned} \overline{X} &= \text{Bl}_S(V) = \{ ([\mathbf{z}], [u, v]) \in W \times \mathbb{CP}^1 \mid f_1(\mathbf{z}) = \dots = f_{k-1}(\mathbf{z}) = 0, v f_k(\mathbf{z}) = u f_{k+1}(\mathbf{z}) \}, \\ \varpi : \overline{X} &\dashrightarrow V, \quad ([\mathbf{z}], [u, v]) \mapsto [\mathbf{z}], \\ E &= \varpi^{-1}(S) = \{ ([\mathbf{z}], [u, v]) \in W \times \mathbb{CP}^1 \mid f_1(\mathbf{z}) = \dots = f_{k+1}(\mathbf{z}) = 0 \} \cong S \times \mathbb{CP}^1, \\ D' &= \overline{\varpi^{-1}(D \setminus S)} = \{ ([\mathbf{z}], [u, v]) \in W \times \mathbb{CP}^1 \mid f_1(\mathbf{z}) = \dots = f_k(\mathbf{z}) = u = 0 \} \\ &= D \times \{ [0, 1] \in \mathbb{CP}^1 \} \cong D, \\ E \cap D' &= S \times \{ [0, 1] \in \mathbb{CP}^1 \} \cong S. \end{aligned}$$

Note that the above equation $v f_k(\mathbf{z}) = u f_{k+1}(\mathbf{z})$ is well-defined because both $f_k(\mathbf{z})$ and $f_{k+1}(\mathbf{z})$ are sections of the line bundle $\mathcal{O}_W(d_k)$. Also, we can compute as

$$\begin{aligned} D' &= \varpi^* D - E, \\ K_{\overline{X}} &= \varpi^* K_V + E = \varpi^*(K_V + D) - D' = -D', \\ N_{D'/\overline{X}} &= D'|_{D'} = D' \cdot D' = 0. \end{aligned}$$

Let $\mathbf{z}' = (z'_0, \dots, z'_{k+3})$ and consider the transformation

$$z'_i = z_i \quad \text{for } i = 0, 1, \dots, k+1, \quad z'_{k+2} = f_k(\mathbf{z}) \quad \text{and} \quad z'_{k+3} = f_{k+1}(\mathbf{z}).$$

Then \mathbf{z}' define well-defined coordinates on W , and we can rewrite \overline{X} and D' as

$$\begin{aligned} \overline{X} &= \{ ([\mathbf{z}'], [u, v]) \in W \times \mathbb{CP}^1 \mid f'_1(\mathbf{z}') = \dots = f'_{k-1}(\mathbf{z}') = 0, v z'_{k+2} = u z'_{k+3} \}, \\ D' &= \{ ([\mathbf{z}'], [u, v]) \in \overline{X} \mid u = 0 \}, \end{aligned}$$

where $f'_i(\mathbf{z}') = f_i(\mathbf{z})$ for $i = 1, \dots, k-1$. In this coordinate system, it follows from the proof of Proposition 3.10 that

$$\tilde{\sigma}(\mathbf{z}', [u, v]) = (\sigma(\mathbf{z}'), [\overline{u}, \overline{v}]) \quad \text{for } (\mathbf{z}', [u, v]) \in \overline{X}.$$

Thus we may assume that the defining function u of D' on \overline{X} satisfies (3.5), so that $\tilde{\sigma}$ is a compatible antiholomorphic involution on \overline{X} .

Now if $k \geq 2$ and we exchange f_k and f_{k-1} and correspondingly choose another f_{k+1} in the above situation, then V and \overline{X} may change, but D and the asymptotic model of $\overline{X} \setminus D$ do not change. Let (\overline{X}_1, D_1) and σ_1 be the former orbifold admissible pair (\overline{X}, D') and $\tilde{\sigma}$, and $(\overline{X}_2, D_2), \sigma_2$ the latter. Setting the isomorphism $\tilde{f} : D_1 \rightarrow D_2$ by

$$\tilde{f} = (\varpi_2|_{D_2})^{-1} \circ \text{id}_D \circ \varpi_1|_{D_1} : D_1 \rightarrow D \rightarrow D_2,$$

we have $\tilde{f} \circ \sigma_1|_{D_1} = \sigma_2|_{D_2} \circ \tilde{f}$. Consequently we have the following

Theorem 4.9. *The above isomorphism \tilde{f} satisfies the gluing condition given in Section 3.4.1. Thus we can apply Theorem 3.24 to $(\overline{X}_i, D_i), \sigma_i$ for $i = 1, 2$, to obtain a compact simply-connected Riemannian 8-manifold M , which has holonomy $\text{Spin}(7)$ if $\hat{A}(M) = 1$.*

4.3. A simple example. In this subsection, we will find a simple example of compact simply-connected 8-manifold M admits a Riemannian metric with holonomy $\text{Spin}(7)$ constructed by Corollary 3.25. We will use the same notation as in Section 4.2.

4.3.1. Setup. Let $W = \mathbb{CP}^4(1, 1, 1, 1, 4)$ be the weighted projective space and $[\mathbf{z}] = [z_0, \dots, z_4]$ be weighted homogeneous coordinates on W , with $\deg z_i = 1$ for $0 \leq i \leq 3$ and $\deg z_4 = 4$. Then W has an isolated singular point $p = [0, 0, 0, 0, 1]$, which is modelled on $\mathbb{C}^4/\mathbb{Z}_4$. If we define an holomorphic involution σ on W by

$$[z_0, z_1, z_2, z_3, z_4] \mapsto [-\overline{z}_1, \overline{z}_0, -\overline{z}_3, \overline{z}_2, \overline{z}_4],$$

then we have $W^\sigma = \{p\} = \text{Sing } W$. Define

$$(4.3) \quad V = W, \quad D = \{ [\mathbf{z}] \in W \mid f_1(\mathbf{z}) = 0 \} \quad \text{and} \quad S = \{ [\mathbf{z}] \in W \mid f_1(\mathbf{z}) = f_2(\mathbf{z}) = 0 \}$$

by weighted homogeneous polynomials

$$(4.4) \quad f_1(\mathbf{z}) = z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 \quad \text{and} \quad f_2(\mathbf{z}) = az_0^8 + az_1^8 + bz_2^8 + bz_3^8 + cz_4^2,$$

where a, b and c are real coefficients. Then we see that conditions (1)–(3), (5) and (6) in Section 4.2 hold. Also, we can choose a, b and c so that condition (4) holds. Thus following Section 4.2, we have an orbifold admissible pair (\overline{X}, D) from V, D and S , where $\overline{X} = \text{Bl}_S(V)$ and we denote the proper transform D' of D by D again. Also, the lift of σ on \overline{X} , which exists by Proposition 3.10 and is denoted by σ again, satisfies conditions (f) and (g) in Definition 3.11, so that σ is a compatible antiholomorphic involution on \overline{X} . Applying the doubling construction in Corollary 3.25, we can resolve the orbifold $M^\nabla = X/\langle\sigma\rangle \cup X/\langle\sigma\rangle$ to obtain a compact 8-manifold M . Then we have the following result.

Theorem 4.10. *This simply-connected 8-manifold M admits a Riemannian metric with holonomy $\text{Spin}(7)$. Moreover M has*

$$\begin{cases} b^2(M) = b^3(M) = 0, \\ b^4(M) = 1678, \\ \chi(M) = 1680 \end{cases} \quad \text{and} \quad \tau(M) = 576.$$

We will show this theorem in Section 4.3.5.

4.3.2. Contributions from the singular point. First, we observe that the branched covering of the isolated singular point $p = [0, 0, 0, 0, 1]$ in $V = \mathbb{CP}^4(1, 1, 1, 1, 4)$. Consider the surjective morphism

$$\pi : \mathbb{CP}^4 \longrightarrow V$$

defined in (4.2), and let $[\mathbf{w}] = [w_0, \dots, w_4]$ be the standard homogeneous coordinates on \mathbb{CP}^4 . Then the restriction of the map π to $\tilde{\Sigma}_4 := \{[\mathbf{w}] \in \mathbb{CP}^4 \mid w_4 = 0\}$ is bijective since $\tilde{\Sigma}_4$ can be identified with \mathbb{CP}^3 . On the other hand, the restriction of the map π to $\tilde{U}_p := \{[\mathbf{w}] \in \mathbb{CP}^4 \mid w_4 \neq 0\} \cong \mathbb{C}^4$ is 4 : 1 except at p . This is because we have $U_p := \{[\mathbf{z}] \in V \mid z_4 \neq 0\} \cong \mathbb{C}^4/\mathbb{Z}_4$ as seen in Section 4.1.1:

$$(4.5) \quad \begin{array}{ccccc} \mathbb{CP}^4 & = & (\tilde{\Sigma}_4 \sqcup \{p\}) & \sqcup & (\tilde{U}_p \setminus \{p\}) \\ \downarrow \pi & & \downarrow 1:1 & & \downarrow 4:1 \\ V & = & (\Sigma_4 \sqcup \{p\}) & \sqcup & (U_p \setminus \{p\}). \end{array}$$

Here we denote $\Sigma_4 = \pi(\tilde{\Sigma}_4) = \{[\mathbf{z}] \in V \mid z_4 = 0\}$.

Next we prove the following result.

Lemma 4.11. *Let \tilde{F} be a projective subvariety of \mathbb{CP}^4 with $\tilde{F} \cap \{p\} = \emptyset$, and $F = \pi(\tilde{F})$. Then we have*

$$\chi(F) = \frac{1}{4}(\chi(\tilde{F}) + 3\chi(\tilde{F} \cap \tilde{\Sigma}_4)).$$

Proof. The property of the map (4.5) yields

$$\begin{aligned} \chi(\tilde{F}) &= \chi(\tilde{F} \setminus \tilde{\Sigma}_4) + \chi(\tilde{F} \cap \tilde{\Sigma}_4) \\ &= 4\chi(F \setminus \Sigma_4) + \chi(F \cap \Sigma_4) \\ &= 4\chi(F) - 3\chi(F \cap \Sigma_4) = 4\chi(F) - 3\chi(\tilde{F} \cap \tilde{\Sigma}_4), \end{aligned}$$

where we used $\tilde{F} \cap \tilde{\Sigma}_4 \cong F \cap \Sigma_4$ for the second and last equalities. Then arrangement shows our result. \square

4.3.3. Computing the cohomology of D . In order to prove Theorem 4.10 first we need to calculate the Euler characteristic $\chi(D)$ of the smooth Calabi-Yau divisor D . We will find this by following two ways.

Computing $\chi(D)$: First way. Let f_1 and f_2 be the weighted homogeneous polynomial defined in (4.4). Then $\tilde{f}_i = \pi^* f_i$ for $i = 1, 2$ are homogeneous polynomials of degree 8 in $\mathbb{C}[w_0, \dots, w_4]$ given by

$$(4.6) \quad \tilde{f}_1(\mathbf{w}) = w_0^8 + w_1^8 + w_2^8 + w_3^8 + w_4^8, \quad \text{and} \quad \tilde{f}_2(\mathbf{w}) = aw_0^8 + aw_1^8 + bw_2^8 + bw_3^8 + cw_4^8,$$

where $[\mathbf{w}] = [w_0, \dots, w_4]$ are the standard homogeneous coordinates on \mathbb{CP}^4 . Setting

$$(4.7) \quad \tilde{D} = \{[\mathbf{w}] \in \mathbb{CP}^4 \mid \tilde{f}_1(\mathbf{w}) = 0\} \quad \text{and} \quad \tilde{S} = \{[\mathbf{w}] \in \mathbb{CP}^4 \mid \tilde{f}_1(\mathbf{w}) = \tilde{f}_2(\mathbf{w}) = 0\},$$

we have $\pi(\tilde{D}) = D$, $\pi(\tilde{S}) = S$ and $\tilde{D} \cap \{p\} = \tilde{S} \cap \{p\} = \emptyset$, so that the assumption of Lemma 4.11 holds for $\tilde{F} = \tilde{D}, \tilde{S}$. Thus $\chi(D)$ is computed in terms of $\chi(\tilde{D})$ and $\chi(\tilde{D} \cap \tilde{\Sigma}_4)$. Since $\tilde{D} \cap \tilde{\Sigma}_4$ is given by

$$(4.8) \quad \tilde{D} \cap \tilde{\Sigma}_4 = \left\{ [\mathbf{w}] \in \mathbb{C}P^4 \mid \tilde{f}_1(\mathbf{w}) = w_4 = 0 \right\} \cong \left\{ [\mathbf{w}'] \in \mathbb{C}P^3 \mid w_0^8 + w_1^8 + w_2^8 + w_3^8 = 0 \right\},$$

where $[\mathbf{w}'] = [w_0, w_1, w_2, w_3]$ are the standard homogeneous coordinates on $\mathbb{C}P^3$, computing the total Chern classes gives

$$\chi(\tilde{D}) = -2096 \quad \text{and} \quad \chi(\tilde{D} \cap \tilde{\Sigma}_4) = 7808,$$

which leads to the following result by Lemma 4.11.

Proposition 4.12. *This smooth Calabi-Yau divisor D on V has the Euler characteristic*

$$\chi(D) = -296.$$

Computing $\chi(D)$: Second way. Theorem 4.7 determines the Hodge numbers of D as follows. Let $R(f)$ be the Jacobian ring of f

$$R(f) = \frac{\mathbb{C}[z_0, \dots, z_4]}{\langle z_0^7, z_1^7, z_2^7, z_3^7, z_4^7 \rangle}.$$

Assume that a graded ring B is finitely generated over \mathbb{C} . Then the *Hilbert series* of the graded ring $B = \bigoplus_m B_m$ is defined to be

$$H_B(t) = \sum_{m=0}^{\infty} (\dim_{\mathbb{C}} B_m) t^m.$$

On the one hand, we can apply [1], Proposition 23.4 to the Jacobian ring $R(f)$. Consequently, the Hilbert series of $R(f)$ is the power series expansion at $t = 0$ of a rational function

$$H_{R(f)}(t) = \frac{(1-t^7)^4}{(1-t)^4} = 1 + 4t + 10t^2 + \dots + 149t^8 + \mathcal{O}(t^9).$$

Then Theorem 4.7 gives

$$h^{3,0}(D) = \dim_{\mathbb{C}} R(f)_0 = 1 \quad \text{and} \quad h^{2,1}(D) = \dim_{\mathbb{C}} R(f)_8 = 149.$$

Thus the Hodge numbers of D are

$$h^{p,q}(D) = \begin{matrix} & & & 1 & & & \\ & & & 0 & & 0 & \\ & 0 & & 1 & & 0 & \\ 1 & 149 & & 149 & & 1 & \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}, \quad \begin{array}{c|c} i & b^i: \text{Betti numbers} \\ \hline 0 & 1 \\ 1 & 0 \\ 2 & 1 \\ 3 & 300 \\ 4 & 1 \\ 5 & 0 \\ 6 & 1 \end{array}$$

Since the Euler characteristic $\chi(D)$ is also given by $\chi(D) = \sum_{p,q} (-1)^{p+q} h^{p,q}(D)$, the result is consistent with Proposition 4.12.

Remark 4.13. Theorem 4.7 is not essential to calculate the Hodge numbers in this example. In fact, we already know that $h^{0,0} = h^{3,0} = 1$ since D is a Calabi-Yau threefold (see [11], Proposition 6.2.6). Therefore the Lefschetz hyperplane theorem and the Euler characteristic determine the Hodge numbers in this case.

4.3.4. Computing the cohomology of S . Analogously to Section 4.3.3, we shall find all Hodge numbers of the weighted complete intersection S defined in (4.3).

Recall that $f_i(\mathbf{z})$ and $\tilde{f}_i(\mathbf{w})$ for $i = 1, 2$ are weighted homogeneous polynomials in $\mathbb{C}[z_0, \dots, z_4]$ and $\mathbb{C}[w_0, \dots, w_4]$ given by (4.4) and (4.6) respectively. Also recall that the smooth complex surface \tilde{S} is a complete intersection given in (4.6) and (4.7), for which we have $\chi(\tilde{S}) = 7808$. As in (4.8), we have

$$\begin{aligned} \tilde{S} \cap \tilde{\Sigma}_4 &= \left\{ [\mathbf{w}] \in \mathbb{C}P^4 \mid \tilde{f}_1(\mathbf{w}) = \tilde{f}_2(\mathbf{w}) = w_4 = 0 \right\} \\ &\cong \left\{ [\mathbf{w}'] \in \mathbb{C}P^3 \mid w_0^8 + w_1^8 + w_2^8 + w_3^8 = aw_0^8 + aw_1^8 + bw_2^8 + bw_3^8 = 0 \right\}, \end{aligned}$$

which is a smooth complex curve in \tilde{S} with $\chi(\tilde{S} \cap \tilde{\Sigma}_4) = -768$. Again by using Lemma 4.11, we find

$$\chi(S) = 1376.$$

Also, we have $b^1(S) = 0$ by the Lefschetz hyperplane theorem. Let us consider the residue ring

$$A = \frac{\mathbb{C}[z_0, \dots, z_4]}{\langle f_1, f_2 \rangle}.$$

Using [1], Proposition 23.4 again we find that the Hilbert series of A can be written as

$$H_A(t) = \frac{(1-t^8)^2}{(1-t)^4(1-t^4)} = 1 + 4t + 10t^2 + \dots + 199t^8 + \mathcal{O}(t^9).$$

Applying Lemma 4.6 to the residue ring A for $q = 2, m = 0$ and $\alpha = 8$, we have

$$h^{0,2}(S) = \dim_{\mathbb{C}} A_8 = 199.$$

Then the Hodge numbers of S are

| | | | | | |
|----------------|-----|-----|-----|-----|-----------------------|
| | 1 | | | i | b^i : Betti numbers |
| | | | | 0 | 1 |
| $h^{p,q}(S) =$ | 0 | 0 | | 1 | 0 |
| | 199 | 976 | 199 | 2 | 1374 |
| | 0 | 0 | | 3 | 0 |
| | | 1 | | 4 | 1 |

since $\chi(S) = 1376$. By the Hodge index theorem, we find the signature of S is

$$\tau(S) = \sum_{p,q=0}^{\dim_{\mathbb{C}} S} (-1)^q h^{p,q} = -576.$$

Summing up our argument in Section 4.3.4, we have the following result.

Proposition 4.14. *This smooth compact complex surface S has*

$$\chi(S) = 1376 \quad \text{and} \quad \tau(S) = -576.$$

4.3.5. Conclusion of the above example. Now we are ready to prove Theorem 4.10. We use the same notation as in Section 4.3.1. The reader should refer to Section 4.4 in [7] as needed.

Proof of Theorem 4.10. First we will compute the Euler characteristic and the signature of the resulting compact simply-connected 8-manifold M . Recall $\varpi : \overline{X} \dashrightarrow V$ is the blow-up of V along the submanifold S . It is well-known that the Euler characteristic of \overline{X} satisfies the equality

$$(4.9) \quad \chi(\overline{X}) = \chi(V) + \chi(E) - \chi(S)$$

where E is the exceptional divisor of the blow-up ϖ . As seen in Section 4.2, we have $E \cong S \times \mathbb{C}P^1$, and so

$$\chi(\overline{X}) = \chi(V) + \chi(S) = 1381,$$

where we used Proposition 4.14 and $\chi(V) = \chi(\mathbb{C}P^4(1, 1, 1, 1, 4)) = 5$. Thus $\chi(X) = \chi(\overline{X}) - \chi(D) = 1677$. Since σ fixes the singular point p in X , we have

$$\chi(X/\langle \sigma \rangle) = \frac{1}{2}(\chi(X) + 1) = 839.$$

Now we construct M by resolving the orbifold $M^\nabla = X/\langle \sigma \rangle \cup X/\langle \sigma \rangle$ with two isolated singular points. Observing from (3.2) that replacing the neighborhood of each singular point in M^∇ with an ALE manifold \mathcal{X}_s adds 1 to the Euler characteristic, we have

$$\chi(M) = \chi(M^\nabla) + 2 = 2\chi(X/\langle \sigma \rangle) + 2 = 1680.$$

To find the signature $\tau(M)$, we see that

$$\tau(\overline{X}) = \tau(V) - \tau(S) = 577$$

in the same manner as (4.9). Hence

$$\begin{aligned}\tau(M^\nabla) &= 2\tau(X/\langle\sigma\rangle) = \tau(X) + 1 \\ &= \frac{1}{2}(2\tau(\overline{X}) - \tau(D \times \mathbb{C}P^1)) + 1 = 578.\end{aligned}$$

Consequently we obtain our result

$$\tau(M) = \tau(M^\nabla) - 2 = 576$$

by taking resolutions of isolated singular points.

Next we shall show that M admits a Riemannian metric with holonomy $\text{Spin}(7)$. This is a consequence of Theorem 2.8. In fact, the Euler characteristic $\chi(M) = 1680$ and the signature $\tau(M) = 576$ above give the \hat{A} -genus $\hat{A}(M) = 1$ for our example. Hence the assertion is verified.

Finally we find the Betti numbers of our $\text{Spin}(7)$ -manifold M . Consider

$$M^\nabla = Z_1 \cup Z_2$$

where $Z_i = X/\langle\sigma\rangle$ for $i = 1, 2$. Then we have homotopy equivalences

$$(4.10) \quad M^\nabla \sim Z_1 \cup Z_2, \quad Z_1 \cap Z_2 \sim (D \times S^1)/\langle\sigma_{D \times S^1, \text{cyl}}\rangle =: Y$$

as in [6], equation (4.6). Here the action of $\sigma_{D \times S^1, \text{cyl}}$ is given by (3.8).

Lemma 4.15 (Kovalev [13]). *Let Z_i ($i = 1, 2$) and Y be as above. Then we have*

$$b^1(Y) = b^2(Y) = 0 \quad \text{and} \quad b^2(Z_i) = b^3(Z_i) = 0.$$

Once Lemma 4.15 has been proved, we conclude that

$$b^2(M^\nabla) = b^3(M^\nabla) = 0$$

by applying the Mayer-Vietoris theorem to (4.10). Then it follows from $\chi(M^\nabla) = 1678$ that

$$b^4(M^\nabla) = 1676.$$

By (15.10) in [11], the Betti numbers $b^j(M)$ satisfy

$$b^j(M) = b^j(M^\nabla) \quad \text{for } j = 1, 2, 3 \quad \text{and} \quad b^4(M) = b^4(M^\nabla) + k$$

where k is the number of singular points in M^∇ . Thus, we conclude our $\text{Spin}(7)$ -manifold M has the Betti numbers $(b^2, b^3, b^4) = (0, 0, 1678)$. In particular, this is a *new* compact $\text{Spin}(7)$ -manifold since no example with the same Betti numbers can be found among the known ones, all of which are listed in [11] and [3].

In order to prove Lemma 4.15, it suffices to show $b^2(Z_i) = b^3(Z_i) = 0$ since $b^j(Y) = 0$ for $j = 1, 2$ were already proved in [13], Proposition 6.2. Recall that $b^2(V) = 1$ and $b^3(V) = 0$ because V is $\mathbb{C}P^4(1, 1, 1, 1, 4)$. Now $\varpi^{-1}(S) \cong S \times \mathbb{C}P^1$ where $\varpi : \overline{X} \dashrightarrow V$ is the blow-up of V along S . Then the Betti numbers $b^i(\overline{X})$ are given by the formula

$$b^i(\overline{X}) = b^i(V) + b^{i-2}(S)$$

(see [4], (1.10)). This gives

$$b^2(\overline{X}) = b^2(V) + b^0(S) = 2 \quad \text{and} \quad b^3(\overline{X}) = b^3(V) + b^1(S) = 0.$$

Since there is a tubular neighborhood U of D in \overline{X} such that

$$(4.11) \quad \overline{X} = X \cup U \quad \text{and} \quad X \cap U \simeq D \times S^1 \times \mathbb{R}_{>0},$$

we apply the Mayer-Vietoris theorem to (4.11). Then we see that

$$(4.12) \quad \begin{cases} b^2(\overline{X}) = b^2(X) + 1, \\ b^3(X) = b^3(\overline{X}) + b^2(D) - b^2(X) \end{cases}$$

(see [14], (2.10)). Let $b^i(X)^\sigma$ be the dimension of the σ -invariant part of $H^i(X, \mathbb{R})$. Then

$$b^2(Z_i) = b^2(X)^\sigma = 0$$

because $H^2(X, \mathbb{R})$ is generated by the Kähler form on X and *not* σ -invariant. Similarly,

$$b^3(Z_i) = b^3(X)^\sigma = 0$$

by (4.12) and the assertion is verified. \square

5. OTHER EXAMPLES

5.1. Complete intersections in $\mathbb{C}P^5(1, 1, 1, 1, 4, 4)$. There are other examples based on the weighted complete intersection of two weighted hypersurfaces in $W = \mathbb{C}P^5(1, 1, 1, 1, 4, 4)$ with homogeneous coordinates $[z] = [z_0, \dots, z_5]$, where $\deg z_i = 1$ for $0 \leq i \leq 3$ and $\deg z_j = 4$ for $j = 4, 5$. Define an antiholomorphic involution $\sigma : W \rightarrow W$ by

$$[z_0, \dots, z_5] \mapsto [-\bar{z}_1, \bar{z}_0, -\bar{z}_3, \bar{z}_2, \bar{z}_4, \bar{z}_5].$$

Consider complete intersections

$$\begin{aligned} V_1 &= \{ [z] \in W \mid f_1(z) = 0 \}, & D_1 &= \{ [z] \in W \mid f_1(z) = f_2(z) = 0 \} & \text{and} \\ S_1 &= \{ [z] \in W \mid f_1(z) = f_2(z) = f_3(z) = 0 \}, \end{aligned}$$

where f_1 and f_2 are defined by

$$f_1(z) = z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^2 - z_5^2 \quad \text{and} \quad f_2(z) = z_0^4 + z_1^4 + z_2^4 + z_3^4 + 2z_4 + z_5,$$

and $f_3(z)$ is chosen so that $\deg f_3 = \deg f_2 = 4$, $\sigma^* f_3 = \overline{f_3}$, and S_1 is a smooth complete intersection in W . Then V_1 has two isolated singular points $p_1 = [0, 0, 0, 0, 1, 1]$ and $p_2 = [0, 0, 0, 0, 1, -1]$, which are modelled on $\mathbb{C}^4/\mathbb{Z}_4$ and fixed by σ . We can see easily that conditions (1)–(6) in Section 4.2 hold, and thus following the argument in Section 4.2 we obtain an orbifold admissible pair (\overline{X}_1, D_1) with a compatible antiholomorphic involution σ_1 .

Similarly, we set $g_1 = f_2$, $g_2 = f_1$ and

$$\begin{aligned} V_2 &= \{ [z] \in W \mid g_1(z) = 0 \}, & D_2 &= \{ [z] \in W \mid g_1(z) = g_2(z) = 0 \} & \text{and} \\ S_2 &= \{ [z] \in W \mid g_1(z) = g_2(z) = g_3(z) = 0 \}, \end{aligned}$$

where we choose g_3 with $\deg g_3 = \deg g_2 = 8$ so that $\sigma^* g_3 = \overline{g_3}$, and S_2 is a smooth complete intersection. Then V_2 has an isolated singular point $p_3 = [0, 0, 0, 0, 1, -2]$, which is modelled on $\mathbb{C}^4/\mathbb{Z}_4$ and fixed by σ . Conditions (1)–(6) in Section 4.2 also hold in this case, and we obtain another admissible pair (\overline{X}_2, D_2) with σ_2 . Note that $(\overline{X}_i \setminus D_i)/\langle \sigma_i \rangle$ for $i = 1, 2$ have the same asymptotic model, and so can be glued together.

Now we can apply Corollary 3.25 and Theorem 4.9. Setting $Z_i = (\overline{X}_i \setminus D_i)/\langle \sigma_i \rangle$ and $M_{ij}^\nabla = Z_i \cup Z_j$, where $i, j \in \{1, 2\}$, we can resolve orbifolds M_{11}^∇ , M_{12}^∇ and M_{22}^∇ to obtain compact simply-connected 8-manifolds M_{11} , M_{12} and M_{22} respectively. Then we see that $\hat{A}(M_{ij}) = 1$ in each case. Hence we conclude that all resulting manifolds M_{ij} are compact $\text{Spin}(7)$ -manifolds. In particular, the resulting manifold M_{22} has the same Betti numbers as the above $\text{Spin}(7)$ -manifold M in Theorem 4.10. Finally we shall list all Hodge numbers in Table 5.3 which are needed to compute $\chi(M_{ij})$ and $\tau(M_{ij})$.

Remark 5.1. Since our examples M_{11}, M_{12} with $(b^2, b^3, b^4) = (0, 0, 910), (0, 0, 1294)$ in Table 5.4 are already listed in [11], we can not distinguish the topological types of these examples from those in [11].

5.2. From the viewpoint of Calabi-Yau structures. In this subsection, we will give an example of Calabi-Yau fourfolds constructed by Corollary 3.23. Ingredients of this example are exactly the same as the previous $\text{Spin}(7)$ -manifold in Section 4.3, so that V is $\mathbb{C}P^4(1, 1, 1, 1, 4)$, D is a smooth anticanonical Calabi-Yau divisor on V , and S is a smooth complex surface in D representing $D \cdot D$. Let $\varpi : \overline{X} \dashrightarrow V$ be the blow-up of V along S . Let \widehat{D} denote (again) the proper transform of $D \in |-K_V|$ under the blow-up ϖ . Then we consider a ‘nice’ resolution of the isolated singular point in \overline{X} as follows. Recall that a complex algebraic variety Y is said to be *Gorenstein* if the canonical divisor K_Y is a Cartier divisor. Suppose that Y is Gorenstein and $\pi : \widehat{Y} \dashrightarrow Y$ is a resolution of Y . We say π is a *crepant resolution* if $\mathcal{O}(K_{\widehat{Y}}) \cong \pi^*(\mathcal{O}(K_Y))$. Now the above \overline{X} has an isolated singular point which is modelled on $\mathbb{C}^4/\mathbb{Z}_4$. Then toric geometry gives a method for finding all crepant resolutions. See [15], and [11], Section 6.4 for more details. For example, the above $\mathbb{C}^4/\mathbb{Z}_4$ has a unique crepant resolution $\widehat{\mathbb{C}^4/\mathbb{Z}_4}$ which is isomorphic to $K_{\mathbb{C}P^3}$. Let $\overline{\pi} : \widehat{X} \dashrightarrow \overline{X}$ and $\pi : \widehat{V} \dashrightarrow V$ be the crepant resolutions of \overline{X} and V respectively which are given by the above argument. Let \widehat{D} denote the proper transform of $D \in |-K_{\overline{X}}|$ under this resolution $\overline{\pi}$. Then there is an induced map

$\widehat{\omega} : \widehat{X} \dashrightarrow \widehat{V}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\omega}} & \widehat{V} \\ \downarrow \pi: \text{crepant} & & \downarrow \pi: \text{crepant} \\ \overline{X} & \xrightarrow{\overline{\omega}} & V \end{array}$$

Here the vertical maps are crepant resolutions and the horizontal maps are the blow-ups of four-dimensional complex algebraic varieties along the complete intersections. By gluing two copies of $\widehat{X} \setminus \widehat{D}$ along their cylindrical ends, we obtain a smooth compact 8-manifold M .

Proposition 5.2. *The above compact simply-connected smooth 8-manifold M is a Calabi-Yau fourfold.*

Proof. Again by Theorem 2.8, it suffices to see that $\widehat{A}(M) = 2$. Let E be the exceptional divisor of the crepant resolution π , that is,

$$\chi(E) = \chi(\widehat{\mathbb{C}^4/\mathbb{Z}_4}) = \chi(K_{\mathbb{C}P^3}) = 4.$$

Then we have the equalities of the Euler characteristics

$$\begin{cases} \chi(\widehat{X}) = \chi(\overline{X}) - 1 + \chi(E) = 1381 - 1 + 4 = 1384, \\ \chi(\widehat{D}) = \chi(D) = -296. \end{cases}$$

This implies

$$\chi(M) = 2(\chi(\widehat{X}) - \chi(\widehat{D})) = 3360.$$

In order to find the signature $\tau(M)$, we see that $\tau(\widehat{X}) = \tau(\overline{X}) - 1 = 576$. Hence we have

$$\tau(M) = 2\tau(\widehat{X}) - \tau(\widehat{D} \times \mathbb{C}P^1) = 2 \cdot 576 - 0 = 1152.$$

Substituting $\chi(M)$ and $\tau(M)$ into (2.5) we have $\widehat{A}(M) = 2$. □

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TABLE 5.3. The list of the Hodge diamonds

| Index | Weighted hypersurfaces in W | Smooth Calabi-Yau divisors on V_i | Weighted complete intersections in V_i |
|-------|---|---|--|
| i | V_i | $D = D_1 = D_2$ | $S_i \in D_i \cdot D_i$ |
| 1 | $ \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 35 & 232 & 35 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & & 0 & 1 & 0 & & \\ & & 0 & 0 & & & \\ & & & 1 & & & \end{array} $ | $ \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & 0 & 1 & 0 & & \\ 1 & 0 & 149 & 149 & 1 & & \\ & 0 & 0 & 1 & 0 & & \\ & & 0 & 0 & & & \\ & & & 1 & & & \end{array} $ | $ \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & 0 & 1 & 0 & & \\ 35 & 0 & 232 & 0 & 35 & & \\ & 0 & 0 & 0 & & & \\ & & 1 & & & & \end{array} $ |
| 2 | $ \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & & 0 & 1 & 0 & & \\ & & 0 & 0 & & & \\ & & & 1 & & & \end{array} $ | $ \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & 0 & 1 & 0 & & \\ 1 & 0 & 149 & 149 & 1 & & \\ & 0 & 0 & 1 & 0 & & \\ & & 0 & 0 & & & \\ & & & 1 & & & \end{array} $ | $ \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & 0 & 1 & 0 & & \\ 199 & 0 & 976 & 0 & 199 & & \\ & 0 & 0 & 0 & & & \\ & & 1 & & & & \end{array} $ |

TABLE 5.4. The resulting Spin(7)-manifolds in Section 5.1

| The resulting Spin(7)-manifolds M | $\tau(M)$ | $\chi(M)$ | b^4 |
|--|-----------|-----------|-------|
| M_{11} | 320 | 912 | 910 |
| M_{12} | 448 | 1296 | 1294 |
| M_{22} | 576 | 1680 | 1678 |

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